On the integration of resonant Poincaré-Dulac normal forms

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Summary. Given a nonlinear dynamical system in $\mathbf{R}^n$ in Poincaré-Dulac normal form, we associate to it an auxiliary linear system; solutions to the original system are obtained from solutions to the auxiliary one on a certain invariant submanifold, defined by resonance conditions. The auxiliary system is finite dimensional if the spectrum of the linearization of the original system satisfies only a finite number of resonance conditions, as implied e.g.by the Poincaré condition.

Introduction

Normal forms are central to our understanding of nonlinear dynamics around known solutions, in more ways than we could recall here; see e.g. [1, 2].

In this note we will consider normal forms of dynamical systems – or, equivalently, vector fields – in $\mathbf{R}^n$ around a regular singular point (for normal forms in this context, see [1, 2, 6, 10, 14], or the short introductions given e.g. in [3, 8, 9, 13]). We will not consider the problem of transforming a general system to its normal form, but rather assume the system is already in (or has already been transformed into) normal form, and consider the problem of integrating it.

We recall that if the linear part of the system is nonresonant (this and other relevant definitions will be recalled in section 1), the normal form is linear and the problem is not so interesting; we will thus assume the linear part is resonant.

We also recall that in general the equivalence between a system and its normal form is only formal; however, under conditions on the linear part we can guarantee *a priori* this equivalence to be analytical. In particular, this is the case if the spectrum of the linear part lies in a Poincaré domain (see below); we are thus specially interested in this case.

The procedure we propose for the integration of resonant normal forms is very simple; it takes advantage of the fact that the time evolution of resonant nonlinear terms can always be written as a linear automorphism of their set (this is proven below) in order to map the nonlinear system to an auxiliary system of larger dimension, but *linear*.

It should be mentioned that we are implicitely using a "triangular" property of equations in resonant normal form which was already remarked by Dulac [5], and which guarantees integrability also with a classical procedure (Dulac attributes this to Horn and Lyapunov); for related modern ideas see [14]. The procedure proposed here is not simpler than classical ones from the analytic point of view; however, it is well suited to be implemented on a computer. Moreover, it clarifies the geometric roots of the integrability of these normal forms.

Acknowledgements

This work came out of an attempt to apply to resonant normal forms some geometric ideas on integrable systems, exposed to the author in a different context by Giuseppe Marmo a long time ago (see [11] and references therein). Such an attempt was actually triggered by questions posed by Dario Bambusi. I would like to warmly thank both of them, for this and for many other things.

1 Normal forms

In this section we fix notation and collect several definitions, and properties of normal forms, to be used in the following.

1.1 Basic notation

Let us consider a C^∞ function $f : \mathbf{R}^n \to \mathbf{R}^n$ such that $f(0) = 0$, expanded in a series of homogeneous terms $f_k(x)$, where $f_k(ax) = a^{k+1} f_k(x)$ for all real number a. This defines a dynamical system in $V_0 = \mathbf{R}^n$ having a singular point at the origin:

$$\dot{x} \ = \ f(x) \ ; \quad x \in \mathbf{R}^n \ , \ f \in C^\infty(\mathbf{R}^n, \mathbf{R}^n) \ , \ f(0) = 0 \tag{1}$$

which we also rewrite, using the expansion in homogeneous terms and singling out the linear part, as

$$\dot{x} \ = \ Ax \ + \ F(x) \ = \ Ax \ + \ \sum_{k=1}^{\infty} f_k(x) \ . \tag{2}$$

We will assume that the linear part of the system, i.e. the matrix A, is in Jordan form.

Remark 1. Transforming a general matrix to Jordan form can be, in practice, very hard for large n; see [12] for a discussion of this point in the context of normal forms theory. $\odot$

1.2 Vector fields

We consider several vector fields associated to this system and decomposition, to be used in the following; we write ∂_i for $\partial/\partial x^i$.

To the function f (i.e. the full dynamical system) we associate the vector field $X_f = f^i(x)\partial_i$; to the linear part $A = (Df)(0)$ we associate the vector field $X_A = (Ax)^i\partial_i$, and to the nonlinear one $X_F = F^i(x)\partial_i$.

Remark 2. Needless to say, the distinction between linear and nonlinear parts is not intrinsic, but depends on our choice of coordinates; however the linear part and hence the vector field X_A (and other ones associated to it, see below) is invariant under near-identity transformations, such as those considered in the normalization algorithm. Moreover, in this note we will not consider normalizing changes of coordinates. $\odot$

We also consider a vector field associated to the adjoint of the matrix A, $X_\ell = (A^+x)^i\partial_i$; as long as we consider real systems and thus real matrices A, this reduces to $X_\ell = (A^Tx)^i\partial_i$.

As well known the matrix A can be decomposed into a semisimple and a nilpotent part, mutually commuting, which we denote as A_s and A_n; as $[A_s, A_n] = 0$, we also have $[A_s, A] = 0 = [A_n, A]$. To the semisimple part A_s of A is associated the vector field $X_0 = (A_sx)^i\partial_i$; we have

$$[X_0, X_A] \ = \ 0 \ = \ [X_0, X_\ell] \ . \qquad (3)$$

Finally, we recall that a matrix A is said to be normal if it commutes with its adjoint, $[A, A^+] = 0$; obviously this is equivalent to the condition that $[X_A, X_\ell] = 0$. In general this is not satisfied, and Jacobi identity only guarantees that $[X_0, [X_A, X_\ell]] = 0$.

1.3 Resonant vectors and monomials

Let $(\lambda_1, ..., \lambda_n)$ be the eigenvalues of A (we denote by σ their ensemble, i.e. the spectrum of A), and let $(\mathbf{e}_1, ..., \mathbf{e}_n)$ be the basis in $V_0 = \mathbf{R}^n$ with respect to which the x^i coordinates are defined. We will use the multiindex notation

$$x^\mu \ := \ x_1^{\mu_1} ... x_n^{\mu_n} \ . \qquad (4)$$

Then the vector $\mathbf{v}_{(\mu)} := x^\mu \mathbf{e}_r$ is resonant with A (or *resonant* for short) if

$$(\mu \cdot \lambda) \ := \ \sum_{i=1}^n \mu_i \lambda_i \ = \ \lambda_\alpha \quad \text{with} \ \ \mu_i \geq 0 \ , \ \ |\mu| := \sum_{i=1}^n \mu_i \geq 2 \ . \qquad (5)$$

Remark 3. As already recalled, a matrix can be decomposed as $A = A_s + A_n$; obviously, by definition, the eigenvalues of A correspond to the eigenvalues of A_s. Thus, when we define and determine resonant monomials and vectors, we can as well consider the semisimple part of A alone. $\odot$

The space of vectors resonant with (the semisimple part of) A is defined as the linear span of such vectors; we will consider a basis $(\mathbf{v}_1, ..., \mathbf{v}_r)$ in this. Thus F is resonant if and only if $F = c_i \mathbf{v}_i$ for some constants c_i.

Remark 4. We stress that r could be infinite, but will always be finite if A admits a finite number of resonance relations, e.g. if σ satisfies the Poincaré condition. On the other side, if the λ_i satisfy a "master resonance" relation, i.e. $\sum_i \mu_i \lambda_i = 0$ with μ_i as in (5) and $|\mu| > 0$ (notice in this case σ cannot satisfy the Poincaré condition), then there is an infinite number of resonances. $\odot$

A monomial x^μ such that (5) is satisfied for some $r \in (1, ..., n)$ is called a *resonant monomial*[1]. We consider a basis of resonant monomials $\{\phi_1(x), ..., \phi_r(x)\}$, and their linear span; this is a linear vector space V_1 (in the space of scalar polynomials on $\mathbf{R}^n$). We choose a basis $\{\mathbf{p}_1, ..., \mathbf{p}_r\}$ in this. Here $\mathbf{p}_i$ corresponds to $\phi_i(x)$, i.e. the scalar polynomial $\sum_{i=1}^r w^i \phi_i(x)$ is represented in V_1 by the vector $\sum_{i=1}^r w^i \mathbf{p}_i$.

1.4 Normal and seminormal forms

We say (see e.g. [6]) that (2) is in Poincaré-Dulac normal form if the vector fields X_ℓ and X_F commute:

$$[\, X_\ell \, , \, X_F \,] \ = \ 0 \ . \tag{6}$$

This implies that all nonlinear terms are resonant with A (i.e. with A_s); however not all resonant terms will satisfy (6) when $A_n \neq 0$, see e.g. example 3 below.

Notice that in general $[X_\ell, X_A] \neq 0$: thus, unless A is normal, we cannot affirm that X_ℓ (or X_A) commutes with X_f.

However it is easy to see from (6) that for systems (2) in Poincaré-Dulac normal form, both X_A and X_F commute with X_0, and therefore

$$[\, X_0 \, , \, X_f \,] \ = \ 0 \tag{7}$$

When the system satisfies (7) – although (6) is possibly not satisfied – i.e. if F is resonant with A_s, we say that it is in normal form with respect to A_s or, that it is in *seminormal form*.

As well known, starting from any dynamical system (or vector field) of the form (1), we can arrive to a dynamical system (or vector field) in Poincaré-Dulac normal form by means of a sequence (in general, infinite) of near-identity transformation obtained by means of the Poincaré algorithm; these combine into a near-identity transformation H defined by a series which is in general only formal. The same applies for seminormal forms.

However, one can guarantee the convergence of the series on the basis of properties of the spectrum σ of the linear part A. In particular, it was already known to Poincaré and Dulac that convergence is guaranteed if the convex hull

[1] Ma serve o basta considerare i resonant vectors ?

of σ in the complex plane does not include the origin; in this case we say that σ belongs to a Poincaré domain, or that A satisfies the Poincaré condition, or also that σ is a Poincaré spectrum.

It is easy to see – and again was well known to Poincaré and Dulac – that if σ satisfies the Poincaré condition, then only a finite number of resonances is present, i.e. the Poincaré-Dulac normal form is finite (conditions ensuring finiteness of the normal form are discussed in [7]).

Remark 5. Other conditions guaranteeing convergence of H, on the basis of σ and of symmetry properties of the normal form, are also known, see e.g. the review [4]. $\odot$

2 Map to a linear system

In this section we will give a very elementary procedure to associate to a nonlinear system (2) in resonant normal form an auxiliary linear system in a real vector space $V \simeq V_0 \times V_1$ (this can be seen as a trivial bundle $\pi : V \to V_0$ over V_0). In particular, if the normal form is finite then $V = \mathbf{R}^N$ for some finite $N > n$, and this procedure provides a way to explicitely and elementarily integrate the system in normal form.

2.1 General construction

Let V_1, $\phi_i(x)$ and $\mathbf{p}_i$ be as defined in subsection 1.3; we assume for ease of language that r is finite (as mentioned above, this is the case for σ satisfying the Poincaré condition). We consider the real vector space $V = \mathbf{R}^{n+r}$ with basis vectors[2] $(\mathbf{e}_1, ..., \mathbf{e}_n; \mathbf{p}_1, ..., \mathbf{p}_r)$ and coordinates $\xi = (x^1, ..., x^n; w^1, ..., w^r)$.

We consider a dynamical system $\dot{\xi} = \psi(\xi)$ [a vector field $X_\psi = \psi^j(\xi)(\partial/\partial\xi^j)$] in V defined as follows: first we rewrite (1), (2) substituting $w^1, ..., w^r$ for $\phi_1(x), ..., \phi_r(x)$ (this gives the evolution equation for the x's); then we assign time evolution for the w by $dw^i/dt = (\partial\phi_i(x)/\partial x^j)(dx^j/dt)$. Having written these equations, we will now consider the x and w as indipendent quantities.

It is clear that, by construction, the manifold $\mathcal{M} \subset V$ defined by

$$w^i - \phi_i(x^1, ..., x^n) = 0 \quad \forall i = 1, ..., r \tag{8}$$

is invariant under the flow of the system (the associated vector field) we have defined in this way. On this invariant manifold, the auxiliary system is equivalent to the original one.

Remark 6. The $\mathcal{M}$ defined by (8) is an algebraic manifold (since the ϕ are polynomials) and it is tangent in the origin to the linear space $\mathbf{R}^n$ defined by $w^i = 0$, i.e. to $V_0 \subset V$ (since the ϕ are nonlinear functions of the x). It is also obvious from (8) that it is a global section of $\pi : V \to V_0$. $\odot$

The auxiliary evolution equations constructed in this way have several general features in common:

[2] Di nuovo attenzione a considerare resonant monomials oppure vectors !

(1) The evolution equation we obtain for (x, w) is *linear*, i.e. $\dot{\xi} = \Psi(\xi)$ reduces to $\dot{\xi} = B\xi$ (with B a matrix), as we show below.

(2) A little thinking shows that we will actually obtain a "block triangular" evolution equation: the evolution of the w depends only on the w themselves, while that of the x depends on the x and the w together.

(3) The system of ODEs we obtain is also (if the coordinates are properly arranged) triangular in proper sense. This is again a general feature (in general, we can have to pass to complex coordinates in the x space and to have a change of variables in the w space to obtain this, see example 4), as easy to see and again as already known to Dulac.

(4) The eigenvalues of B (see point 1) are given by $(\lambda_1, ..., \lambda_n; \lambda_{\alpha_1}, ..., \lambda_{\alpha_r})$, where the λ_i's are the eigenvalues of A, and α_i is the α associated to the resonant monomials ϕ_i, see (5); thus, we always have multiple eigenvalues. Once again, this is readily seen by direct elementary computation.

2.2 Linearity of the auxiliary system

Needless to say, the most interesting feature among those claimed above for the auxiliary system is its linearity. We will now substantiate the assertion that the system obtained according to the procedure given above will always be linear, leaving the proof of the other properties to the interested reader. We will actually proof thius in two ways, one algebraic and one geometric.

Algebraic proof.

Summation over repeated indices will be understood, and we will denote by $\nu(i)$ the multiindex such that $\nu_j = \delta_{ij}$.

Let us consider a resonant monomial $w = x^\mu$, and say it satisfies $(\lambda \cdot \mu) := \lambda_i \mu_i = \lambda_\alpha$. We denote by $w_{\alpha;i;\sigma}$, $i = 1, ..., q(\alpha)$ the resonant monomials x^σ such that $(\lambda \cdot \sigma) = \lambda_\alpha$ (with the same fixed α). Then we have that (for f in seminormal form) under $\dot{x} = f(x)$ the time evolution of w is given by $X_f(w)$, i.e.

$$\frac{dw}{dt} \ = \ \frac{\partial w}{\partial x^i} \, f^i(x) \ = \ \mu_i x^{\mu - \nu(i)} \ \left[(A_s)^i_{\ j} x^j + (A_n)^i_{\ j} x^j + c_m w_{\alpha;m;\sigma} \right] \ , \tag{9}$$

where we have used the decomposition (2) and the fact all the nonlinear terms must be resonant.

We assume that A is in Jordan form, so that $A_s = \mathrm{diag}(\lambda_1, ..., \lambda_n)$, and $(A_n)^i_{\ j} = \eta_{ij}$ is different from zero (and equal to one) if and only if $j = i + 1$ and x^i, x^j belong to the same Jordan block; this implies of course that $\lambda_i = \lambda_j$.

Notice that terms with $\mu_i = 0$ are absent from the sum over i; we can therefore assume $\mu_i \neq 0$. Under this condition, and using the assumption of A in Jordan form, we can rewrite

$$\dot{w} \ = \ (\lambda \cdot \mu) x^\mu \ + \ \mu_i \eta_{ij} x^{\mu - \nu(i) + \nu(j)} + c_\sigma \mu_i x^{\mu - \nu(i) + \sigma} \ . \tag{10}$$

The first term on the r.h.s. is nothing else that $\lambda_\alpha w$. We want to check that the other terms are also (the sum of) resonant monomials; in order to do this we do not have to worry about the scalar coefficients in fronts of them.

The monomial appearing in the second term of the r.h.s. are of the form $x^\varphi = x^{\mu-\nu(j)+\nu(k)}$, and we can assume $\mu_i \neq 0$ and $\eta_{ij} \neq 0$ (or the corresponding monomial would not be present in the sum). For this we have

$$(\lambda \cdot \varphi) \ := \ \lambda_i \varphi_i \ = \ (\lambda \cdot \mu) - \lambda_i + \lambda_j \ = \ (\lambda \cdot \mu) \ = \ \lambda_\alpha \ ; \tag{11}$$

we have used the fact that $\eta_{ij} \neq 0$ implies $\lambda_i = \lambda_j$, and the resonance relation satisfied by w itself. Thus the second term in the r.h.s. of (8) is the sum of resonant monomials (with the same α as w).

The monomials appearing in the third term are of the form $x^\varphi = x^{\mu-\nu(i)+\sigma}$, where $(\lambda \cdot \sigma) = \lambda_i$ and we can assume $\mu_i \neq 0$ (or the corresponding monomial would be absent from the sum). We have now

$$(\lambda \cdot \varphi) \ := \ \lambda_i \varphi_i \ = \ (\lambda \cdot \mu) - \lambda_i + (\lambda \cdot \sigma) \ = \ \lambda_\alpha \ ; \tag{12}$$

again we have a sum of resonant monomials (with the same α as w).

This concludes the proof that $\dot{w}$ can be written as a linear combination of resonant monomials, i.e. that the evolution equation constructed according to our procedure is linear. $\triangle$

Remark 7. Notice that we have actually proved something more, i.e. that if $w = x^\mu$ with $(\lambda \cdot \mu) = \lambda_\alpha$, only resonant monomials $\widetilde{w} = x^\pi$ with $(\lambda \cdot \pi) = \lambda_\alpha$ (with the same α as above) will appear in this linear combination. That is, the matrix B will be a block one, where the blocks correspond to resonant monomials identified as described here. $\odot$

Geometric proof

A more geometric (but equivalent) proof could be obtained by considering a basis of (nonlinear) resonant vectors $x^\mu \mathbf{e}_k$ ($k = 1, ..., n$) and the corresponding vector fields $X^{(i)} = (x^\mu)\partial_k$ ($i = 1, ..., r$, see above). These obviously generate a Lie algebra $\mathcal{G}$ (recall that the resonance condition is equivalent to commutation with X_0, and notice that the commutator of two vectors built from nonlinear monomial terms will never be a linear vector field).

One should then check that $\mathcal{G}$ is globally invariant under time evolution, i.e. that $[X_f, \mathcal{G}] \subseteq \mathcal{G}$.

Take $X_\phi \in \mathcal{G}$: we have $[X_f, X_\phi] = [X_A, X_\phi] + [X_F, X_\phi]$, and the second term is by definition in $\mathcal{G}$. To see that the first is also in $\mathcal{G}$, we have to check the vanishing of $[X_0, [X_A, X_\phi]]$; using the Jacobi identity, this is $[X_A, [X_0, X_\phi]] - [X_\phi, [X_0, X_A]]$, and both terms vanish separately. The proof is complete. $\triangle$

Remark 8. Notice that if A is not normal, the set of resonant monomials defining vectors in normal form with respect to A^+ would not, in general, be closed under time evolution. This is also immediately seen from the alternative geometrical proof by remarking that if we substitute X_ℓ for X_0, we have (for A not normal) $[X_\ell, X_A] \neq 0$. Thus for non-normal A we cannot limit to consider vectors in normal form, but have to consider the set of all resonant vectors. This will also be clearly shown in example 3 below. $\odot$

2.3 Truncated normal forms

When the normalizing transformation is not convergent, the normal form is not conjugated to the original system and therefore the study of normal forms does not give information on the original system. However, in such a case one can consider normalization only up to a sufficiently low degree N (in practice this is determined by either the computational limits or the optimal degree on the basis of convergence properties of the truncated series); in this way one obtains a system which is in normal form up to order N. This system can be truncated at order N – thus obtaining a truncated normal form – and the relation of such a truncation with the full system studied via other techniques in perturbation theory [1, 2, 8, 13].

The block structure of the B matrix, as determined above (see in particular remark 7), explains when this truncation will result in a closed auxiliary system.

Indeed, consider all the resonance relations (5); let $m_-(\alpha)$ and $m_+(\alpha)$ be the smaller and greater values of $|\mu|$ for which a relation $(\mu \cdot \lambda) = \lambda_\alpha$ is satisfied. It follows from remark 7 that if

$$N \notin [m_-(\alpha), m_+(\alpha)] \quad \forall \alpha = 1, ..., n \ , \tag{13}$$

then the truncated normal form at order N is mapped, by the procedure discussed in this note, to a closed linear system.

Notice that for a finite dimensional system we always have a finite number of resonances with $|\mu| \leq N$ (for N finite). Thus when we truncate a normal form at a finite order N, we have only a finite number of resonant monomials present in the truncated normal form; if (13) is satisfied, we have an auxiliary finite dimensional linear system, which is readily integrated to give exact solutions to the truncated normal form equations (see below).

Thus the integration procedure discussed here is also of use in cases where the normalizing transformation is not convergent in any neighbourhood of the origin.

3 Integration of normal forms

3.1 Integration via the auxiliary system

The strategy to integrate normal forms via the auxiliary linear system $\dot{\xi} = B\xi$ we have defined is rather obvious: this rests on the dynamical invariance of the manifold defined by (8). That is,

1. *Step 1.* For $\xi = (x; w)$, determine the general solution of the linear equation $\dot{\xi} = B\xi$ in $V = \mathbf{R}^{n+r}$, say with solution $\widehat{\xi}(t)$ where $\widehat{\xi}(0) = \xi_0 =$

(x_0, w_0) is the initial datum. This will depend on the $n + r$ arbitrary constants (x_0, w_0).

2. *Step 2.* Restrict the general solution to the invariant n-dimensional submanifold $\mathcal{M} \subset \mathbf{R}^{n+r}$ defined by $w^i = \phi_i(x^1, ..., x^n)$. This will depend on the n arbitrary constants x_0.

3. *Step 3.* Project the general solution $(x(t), w(t))$ on $\mathcal{M} \subset V$ to the subspace $V_0 = \mathbf{R}^n$ spanned by the x variables, i.e. extract $x(t)$ forgetting about $w(t)$.

The correspondence between the original nonlinear system and the restriction of the auxiliary system to the invariant manifold $\mathcal{M}$ is clear by construction, and projection is globally well defined as $\mathcal{M}$ is identified by the algebraic equations (8). It is therefore clear that this procedure will indeed provide the most general solution to the original nonlinear system in V_0.

This strategy will be particularly simple, and successful, when there is only a finite number of resonances, and in particular when σ belongs to a Poincaré domain.

3.2 Relations with other integration methods

It should be stressed that if σ belongs to a Poincaré domain, the normal forms could of course also be integrated directly: indeed the corresponding system is nonlinear but, as remarked by Dulac [5], always in triangular form. Namely, we can always write $\dot{x}^i = A^i_j x^j + \Phi^i(x)$ in such a way that $\partial \Phi^i / \partial x^j = 0$ for $j > i$. It is then possible to solve the equations recursively, starting from the linear one for $x^1(t)$ and having at each step a linear equation with a forcing term which is a nonlinear but explicitely known function of t.

The procedure proposed here is equivalent to the one considered by Dulac (and attributed by him to Horn and Lyapunov) from the analytic point of view, but can be more easily implemented on computers via algebraic manipulation languages, i.e. can be more convenient from the computational point of view.

As pointed out by prof. Walcher after the first version of this note was circulated, the basic ideas behind this approach can also be found in [14]; however I believe the implementation given in this note is somewhat simpler.

It also has the advantage of showing how the nonlinear system is obtained by restriction (on the submanifold $\mathcal{M}$) of a linear system via nonlinear constraints, clarifying the connection with topics in modern integrable systems theory.

4 Examples

Systems in normal form are specially interesting if a small neighbourhood of the origin is dynamically invariant, i.e. if the singular point is stable (in this case the evolution will remain in the domain of analyticity of the normalizing transformation); thus we are mainly interested in cases where the real part of the eigenvalues is negative (or zero). However, for ease of notation we will consider examples with positive eigenvalues; the stable situation is recovered by a time reversal. Also for ease of notation, we will write all vector indices as lower ones; the c_i will be arbitrary real constants.

Example 1.

Let us consider $n = 2$, with coordinates (x, y) and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} ;$$

notice here $\sigma = \{1, k\}$ is in a Poincaré domain. There is only one resonance $\mu = (k, 0)$ (with $\alpha = 2$), and the only resonant monomial is $\phi(x, y) = x^k$. The Poincaré-Dulac normal form is

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= ky + c_1 x^k \end{aligned}$$

with c_1 an arbitrary coefficient.

Thus, following our procedure, we set $\dot{w} = k(x^{k-1})\dot{x} = kw$; the system obtained in V is

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= ky + c_1 w \\ \dot{w} &= kw \end{aligned}$$

That is, we have a linear equation $\dot{\xi} = B\xi$, where $\xi = (x, y, w)$ and

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & c_1 \\ 0 & 0 & k \end{pmatrix}$$

The solution to this is given by

$$x(t) = x_0 e^t \ , \ y(t) = y_0 e^{kt} + (c_1 k w_0) t e^{kt} \ , \ w(t) = w_0 e^{kt} \ ;$$

obviously the submanifold $\mathcal{M}$ identified by $w = x^k$ is invariant under this flow, and the projection of solutions on $\mathcal{M}$ to $\mathbf{R}^2 = (x, y)$ is simply

$$x(t) = x_0 e^t \quad , \quad y(t) = [y_0 + (c_1 k x_0^k) t] \, e^{kt}$$

Example 2.

Let us consider $n = 3$, with coordinates (x, y, z) and

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} ;$$

notice here $\sigma = \{1, 2, 5\}$ is in a Poincaré domain. There are four resonances:

$$\begin{cases} \mu = (2,0,0) & (\text{with } \alpha = 2 \text{ and } |\mu| = 2) \\ \mu = (1,2,0) & (\text{with } \alpha = 3 \text{ and } |\mu| = 3) \\ \mu = (3,1,0) & (\text{with } \alpha = 3 \text{ and } |\mu| = 4) \\ \mu = (5,0,0) & (\text{with } \alpha = 3 \text{ and } |\mu| = 5) \end{cases}$$

and correspondingly we have

$$\phi_1 = x^2 \ , \ \phi_2 = xy^2 \ , \ \phi_3 = x^3 y \ , \ \phi_4 = x^5 \ .$$

The normal form is written as

$$\begin{aligned} \dot{x} &= \ x \\ \dot{y} &= \ 2y + c_1 x^2 \\ \dot{z} &= \ 5z + c_2 xy^2 + c_3 x^3 y + c_4 x^5 \end{aligned}$$

with c_i arbitrary real coefficients.

By our procedure, we obtain a system in $V = \mathbf{R}^7$, given by

$$\begin{aligned} \dot{x} &= \ x \\ \dot{y} &= \ 2y + c_1 w_1 \\ \dot{z} &= \ 5z + c_2 w_2 + c_3 w_3 + c_4 w_4 \\ \dot{w}_1 &= \ 2w_1 \\ \dot{w}_2 &= \ 5w_2 + 2c_1 w_3 \\ \dot{w}_3 &= \ 5w_3 + c_1 w_5 \\ \dot{w}_4 &= \ 5w_4 \end{aligned}$$

Thus the system for $\xi = (x, y, z, w_1, w_2, w_3, w_4)$ is linear, $\dot{\xi} = B\xi$ where

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & c_1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & c_2 & c_3 & c_4 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 2c_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

(notice this has the block structure discussed in section 2). The general solution to this (writing $w_i(0) = \gamma_i$ for ease of notation) is

$$\begin{aligned} & x(t) = x_0 e^t \quad , \quad y(t) = (y_0 + \gamma_1 t) e^{2t} \\ & z(t) = \left[z_0 + (\gamma_2 c_2 + \gamma_3 c_3 + \gamma_4 c_4) t + (1/2)(2\gamma_3 c_1 c_2 + \gamma_4 c_3) t^2 \right] e^{5t} \\ & w^1(t) = \gamma_1 e^{2t} \quad , \quad w^2(t) = (\gamma_2 + 2c_1 \gamma_3 t) e^{5t} \\ & w^3(t) = (\gamma_3 + \gamma_4 t) e^{5t} \quad , \quad w^4(t) = \gamma_4 e^{5t} \end{aligned}$$

which once restricted to the manifold $\mathcal{M}$ (here identified by $w_1 = x^2$, $w_2 = xy^2$, $w_3 = x^3 y$, $w_4 = x^5$) and projected to $\mathbf{R}^n$, gives

$$\begin{array}{rcl} x(t) &=& x_0 \, e^t \\ y(t) &=& (y_0 \, + \, x_0^2 \, t) \, e^{2t} \\ z(t) &=& \left[z_0 \, + \, (c_2 x_0 y_0^2 + c_3 x_0^3 y_0 + c_4 x_0^5) \, t \, + \, (2 c_1 c_2 x_0^3 y_0 + c_3 x_0^5) \, (t^2/2) \right] \; e^{5t} \end{array}$$

Example 3.

Let us consider $n = 3$, with coordinates (x, y, z) and

$$A \; = \; \begin{pmatrix} 1 & \eta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \ ;$$

notice here $\sigma = \{1, 1, 2\}$ is in a Poincaré domain, and $A_n \neq 0$ for $\eta \neq 0$.

There are three resonances, all of them with $\alpha = 3$ and $|\mu| = 2$, i.e.

$$\mu = (2,0,0) \ , \ \mu = (1,1,0) \ , \ \mu = (0,2,0)$$

and correspondingly

$$\phi_1 = x^2 \ , \ \phi_2 = xy \ , \ \phi_3 = y^2 \ .$$

The seminormal form with respect to the semisimple part A_s of A is thus

$$\begin{array}{rl} \dot{x} = & x + \eta x \\ \dot{y} = & y \\ \dot{z} = & 2z + c_1 x^2 + c_2 xy + c_3 \end{array}$$

The normal form with respect to the full A is the same for $\eta = 0$ (which means $A = A_s$), and for $\eta \neq 0$ is obtained setting $c_2 = c_3 = 0$, as readily seen by considering $X_F = (c_1 x^2 + c_2 xy + c_3 y^2) \partial_z$ and imposing $[X_\ell, X_F] = 0$.

The associated linear system in $V = \mathbf{R}^6$ is $\xi = B\xi$, with

$$B \; = \; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \eta & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & c_1 & c_2 & c_3 \\ 0 & 0 & 0 & 2 & \eta & 0 \\ 0 & 0 & 0 & 0 & 2 & \eta \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

for the seminormal form, and the same – with $c_2 = c_3 = 0$ if η is nonzero – when we normalize with respect to the full A.

Writing again $w_i(0) = \gamma_i$, the general solution to this linear system is

$$\begin{array}{l} x(t) = x_0 e^t \\ y(t) = (y_0 + \eta x_0 t) e^t \\ z(t) = \left[z_0 + (c_1 \gamma_1 + c_2 \gamma_2 + c_3 \gamma_3) t + (c_1 \gamma_2 + \eta c_2 \gamma_3)(t^2/2) + \eta c_1 \gamma_3 (t^3/6) \right] e^{2t} \\ w_1(t) = (\gamma_1 + \gamma_2 t + \eta \gamma_3 t^2/2) e^{2t} \\ w_2(t) = (\gamma_2 + \eta \gamma_3 t) e^{2t} \\ w_3(t) = \gamma_3 e^{2t} \end{array}$$

Restricting to $\mathcal{M}$ (which in this case means setting $w_1 = x^2$, $w_2 = xy$, $w_3 = y^2$) and projecting to $\mathbf{R}^3$ gives

$$\begin{array}{rcl} x(t) & = & x_0 \; e^t \\ y(t) & = & (y_0 \; + \; \eta x_0 \, t) \; e^t \\ z(t) & = & [z_0 \; + \; (c_1 x_0^2 + c_2 x_0 y_0 + c_3 y_0^2) \, t \; + \\ & & \; + \; (c_1 x_0 y_0 + \eta c_2 y_0^2) \, (t^2/2) \; + \; \eta c_1 y_0^2 \, (t^3/6)] \; e^{2t} \end{array}$$

If $\eta \neq 0$ and the system is in normal form with respect to the full matrix A (so that $c_2 = c_3 = 0$) then $z(t)$ simplifies to

$$z(t) \;=\; \left[z_0 + c_1 x_0^2 t + c_1 x_0 y_0 (t^2/2) + \eta c_1 y_0^2 (t^3/6) \right] e^{2t} \ .$$

Notice that (for $\eta \neq 0$) the time evolution of ϕ_1 depends on ϕ_2, and through this on ϕ_3 as well; thus, it would not be possible to consider an auxiliary system involving only terms in normal form with respect to A^+. For $\eta = 0$ the equations for the w_i decouple, but then all the ϕ_i are allowed in the normal form.

Example 4.

Let us consider $n = 2$ with coordinates (x, y) and

$$A \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Here $\sigma = \{-i, i\}$ does not satisfy the Poincaré condition. We have a master resonance $\lambda_1 + \lambda_2 = 0$, and hence an infinite number of resonances, given by

$$\begin{cases} \mu = (k+1, k) & \text{i.e. } (k+1)\lambda_1 + k\lambda_2 = \lambda_1 \ , \ \alpha = 1 \\ \mu = (k, k+1) & \text{i.e. } k\lambda_1 + (k+1)\lambda_2 = \lambda_2 \ , \ \alpha = 2 \end{cases}$$

and the resonant monomials are given by

$$\phi_{2m-1} = (x^2 + y^2)^m x = \rho^m x \ , \ \ \phi_{2m} = (x^2 + y^2)^m y = \rho^m y \ ,$$

where $\rho = (x^2 + y^2)$. We also write, for ease of notation, $\chi_m \equiv \phi_{2m-1}$, $\pi_m \equiv \phi_{2m}$, and correspondingly we will introduce coordinates $q_m = w_{2m-1}$, $p_m = w_{2m}$.

The normal form is then

$$\begin{array}{rl} \dot{x} = & -y + \sum_{k=1}^\infty (x^2 + y^2)^k (a_k x - b_k y) \\ \dot{y} = & +x + \sum_{k=1}^\infty (x^2 + y^2)^k (b_k x + a_k y) \end{array}$$

and this is rewritten following our procedure as an infinite linear system:

$$\begin{array}{rl} \dot{x} = & -y + \sum_{k=1}^\infty (a_k q_k - b_k p_k) \\ \dot{y} = & +x + \sum_{k=1}^\infty (b_k q_k + a_k p_k) \\ \dot{q}_k = & -p_k + \sum_{s=1}^\infty \ [(2k+1) a_s q_{k+s} - b_s p_{k+s}] \\ \dot{q}_k = & +q_k \sum_{s=1}^\infty \ [b_s q_{k+s} + (2k+1) a_s p_{k+s}] \end{array}$$

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