On nodal solutions to generalized Emden-Fowler equations

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Abstract

We introduce a new variational method in order to derive results concerning existence and nodal properties of solutions to superlinear equations, and we focus on applications to the equation

$$-\Delta u = h(x,u)$$
$$u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N), \quad \nabla u \in L^2(\mathbb{R}^N), \quad N \geq 3$$

where h is a Caratheodory function which is odd in u. In the particular case where h is radially symmetric, we prove, for given $n \in \mathbb{N}$, the existence of a solution having precisely n nodal domains, whereas some results also pertain to a nonsymmetric nonlinearity.

1 Introduction

We consider the semilinear elliptic equation

$$-\Delta u = f(x,u^2)u \qquad\qquad u \in D^{1,2}(\mathbb{R}^N) \tag{1.1}$$

for $N \geq 3$, and we are interested in solutions with prescribed nodal structure, i.e. with additional information on the set

$$\Lambda_u := \{x \in \mathbb{R}^N \mid u(x) \neq 0\}.$$

A component of Λ_u will be called a *nodal domain* of u. Moreover, we denote $D^{1,2}(\mathbb{R}^N)$ the space of functions $u \in L^{\frac{2N}{N-2}}$ with distributional derivative $\nabla u \in L^2(\mathbb{R}^N)$. Note that by Sobolev's inequality $D^{1,2}(\mathbb{R}^N)$ becomes a Hilbert space with scalar product

$$(u|v) := \int_{\mathbb{R}^N} \nabla u \nabla v.$$

We present results for

(a) a radially symmetric setting, i.e. we assume $f(x,u^2) = \mathfrak{f}(|x|,u^2)$ and we focus on radially symmetric solutions,

(b) a setting without underlying symmetry.

In both cases we assume that $f : \mathbb{R}^N \times [0,\infty[\to \mathbb{R}$ is a Caratheodory function that satisfies

$(\mathcal{F}_1)$ For a.e. $x \in \mathbb{R}^N$ there holds $f(x,0) = 0$, moreover $f(x,\cdot)$ is nondecreasing on $[0,\infty[$ and strictly increasing once it takes positive values.

$(\mathcal{F}_2)$ There is $\eta > 2$ such that $0 \le \frac{\eta}{2}\int_0^{t^2} f(x,s)\,ds \le f(x,t^2)t^2$ for $t \ge 0$.

$(\mathcal{F}_3)$ $f \not\equiv 0$.

In case (a) we assume in addition

(A) There are constants $c_1, c_2 > 0$, $\alpha_1, \alpha_2 > -2$, $\beta_1, \beta_2 > 0$ such that for $s \ge 0$

$$\mathfrak{f}(r,s) \le \begin{cases} c_1 r^{\alpha_1} s^{\beta_1} & \text{for} \quad 0 < r \le 1 \\ c_2 r^{\alpha_2} s^{\beta_2} & \text{for} \quad 1 \le r < \infty \end{cases}$$

and $\alpha_1 - (N-2)\beta_1 > -2$ as well as $\alpha_2 - (N-2)\beta_2 < -2$.

In case (b) we assume

(B) There are constants $a \in (0,2)$ and C, $\gamma > 0$ such that for $s \ge 0$

$$f(x,u) \le C\frac{1}{(1+|x|)^a}|u|^\gamma, \qquad x \in \mathbb{R}^N, \quad u \in \mathbb{R}$$

and

$$\frac{2-a}{N-2} < \gamma < \frac{2}{N-2}.$$

Concerning existence of nodal solutions, we have

Theorem 1.1 *Suppose that $f(x,u^2) = \mathfrak{f}(|x|,u^2)$ and that $(\mathcal{F}_1)$,$(\mathcal{F}_2)$,$(\mathcal{F}_3)$ and (A) hold. Then for each $n \in \mathbb{N}$ there is a radially symmetric (weak) solution $u_n \in D^{1,2}(\mathbb{R}^N)$ of (1.1) having precisely n nodal domains.*
Moreover, $u_n \in C(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{0\}) \cap W^{2,p}(\mathbb{R}^N)$ for some $p > \frac{N}{2}$.

Theorem 1.2 *Suppose that $(\mathcal{F}_1)$,$(\mathcal{F}_2)$,$(\mathcal{F}_3)$ and (B) hold. Then there are (weak) solutions u_1, $u_2 \in D^{1,2}(\mathbb{R}^N)$ of (1.1) such that u_1 is positive and u_2 has precisely two nodal domains. Moreover, u_1, $u_2 \in C^1(\mathbb{R}^N) \cap W^{2,p}(\mathbb{R}^N)$ for $1 \le p < \infty$.*

Remark: (a) Note that condition (B) implies that the nonlinearity is subcritical in u, whereas condition (A) admits supercritical nonlinearities as well.

(b) Obviously the solutions of Theorem 1.1 and Theorem 1.2 are classical provided that the function $(x,u) \to f(x,u^2)u$ is Hölder continuous in (x,u) and locally Lipschitz with respect to u.

One way to derive the existence of infinitely many solutions to (1.1) seems to be the application of Ljusternik-Schnirelman theory on the Nehari type set

$$\mathcal{N} := \left\{ u \in D^{1,2}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |\nabla u|^2 = \int_{\mathbb{R}^N} f(x,u^2)u^2 \right\} - \{0\},$$

(resp. on $\mathcal{N}_R := \{u \in \mathcal{N} \mid u$ radially symmetric $\})$, which is a closed subset of $D^{1,2}(\mathbb{R}^N)$ containing all nontrivial solutions (resp. radial solutions) of (1.1). Note however that under our assumptions $\mathcal{N}$ and $\mathcal{N}_R$ might not be C^1-manifolds. Moreover, classical Ljusternik-Schnirelman-Theory does not give rise to the nodal properties that we assert. Consider the energy functional

$$\psi(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2}\int_{\mathbb{R}^N}\int_0^{u^2(x)} f(x,s)\,ds\,dx \qquad (u \in D^{1,2}(\mathbb{R}^N))$$

corresponding to (1.1) and the Ljusternik-Schnirelman levels

$$c_n := \inf_{A\in\Sigma_n(\mathcal{N})} \sup_{u\in A} \psi(u) \tag{1.2}$$

where $\Sigma_n(\mathcal{N})$ denotes the family of all closed and symmetric subsets $A \subset \mathcal{N}$ with Krasnosel'skii genus $\gamma(A) \geq n$. Similarly one defines $\tilde{c}_n$, replacing $\mathcal{N}$ by $\mathcal{N}_R$ in (1.2).
Surprisingly, under the assumptions stated above, *any* solution of (1.1) on a Ljusternik-Schnirelman level set is characterized by the following observation:

Theorem 1.3 *Let $u \in D^{1,2}(\mathbb{R}^N)$.*

(a) If u is a weak solution of (1.1) such that either $\psi(u) \leq c_n$ or $\psi(u) < c_{n+1}$, then u has at most *n nodal domains.*

(b) If u is a weak radial solution of (1.1) such that either $\psi(u) \leq \tilde{c}_n$ or $\psi(u) < \tilde{c}_{n+1}$, then u has at most *n nodal domains.*

Actually the solutions u_n to which we refer in Theorem 1.1 (resp. Theorem 1.2) will be constructed satisfying

$$\psi(u_n) = \tilde{c}_n \qquad (\text{ resp. } \psi(u_n) = c_n).$$

Hence the following nondegeneracy property of the Ljusternik-Schnirelman levels is immediate:

Corollary 1.4 *$\tilde{c}_n < \tilde{c}_{n+1}$ for every $n \in \mathbb{N}$, and $c_1 < c_2$. Moreover, c_2 (resp. $\tilde{c}_2$) is the minimal 'energy' of sign changing weak solutions (resp. sign changing radial solutions).*

Some comments are in order. First note that NAITO [19] and CHABROWSKI [6] have considered the existence of radial nodal solutions to

$$-\Delta u - q(|x|)h(u) = 0 \qquad (u \in D^{1,2}(\mathbb{R}^N)). \tag{1.3}$$

In [19] the case $h(u) = |u|^{\beta-1}u$ is considered, where $\beta > 1$ and

$$q \in C[0,\infty) \cap C^1(0,\infty) \quad q(r) > 0 \text{ for } r > 0 \tag{1.4}$$

as well as

$$\liminf_{r\to 0} \frac{rq'(r)}{q(r)} > -\frac{N+2-\beta(N-2)}{2} \tag{1.5}$$

$$\limsup_{r\to\infty} \frac{rq'(r)}{q(r)} < -\frac{N+2-\beta(N-2)}{2}. \tag{1.6}$$

Using ODE-techniques, NAITO showed that if (1.4)-(1.6) hold, then for given $n \in \mathbb{N}$ there exists a radial solution u with precisely n nodal domains (cf. [19, Theorem 7]). Note that (1.4)-(1.6) imply $(\mathcal{F}_1)$, $(\mathcal{F}_2)$, $(\mathcal{F}_3)$ and (A). On the other hand, if a nonlinearity of the form $q(|x|)|u|^{\beta-1}u$, $\beta > 1$ satisfies (A), we only require

$$q \in L^\infty_{loc}(0,\infty)\ , q \geq 0,\ q \not\equiv 0 \tag{1.7}$$

instead of (1.4) to ensure our remaining assumptions. In particular we cover cases where q is singular at the origin. Moreover, in view of condition $(\mathcal{F}_3)$, $\operatorname{supp} q$ may be contained in a very small interval $I \subset (0,\infty)$, and in this case we conclude that the nodal solutions $u_n = u_n(r)$ found by Theorem 1.1 only change sign in I. This follows since u_n satisfies

$$(r^{N-1}u_n')' = 0, \quad u_n'(0) = 0 \quad \lim_{r\to\infty} u_n(r) = 0,$$

on $[0,\infty) \setminus I$, hence it can not change sign in this region. Observe furthermore that our assumptions also admit nonhomogeneous nonlinearities h in (1.3). These are also considered in [6], but there it is assumed that h at least is not supercritical, and q has to be continuous on $[0,\infty)$ with $q(r) > 0$ for all positive r as well as $q(0) = 0$ and $\lim_{r\to\infty} q(r) = 0$. Note moreover that the methods of [6] do not carry over to growth conditions of the form (1.7). Nevertheless we also remark that in [6] no oddness of h is assumed, and in [19] also the sublinear case $0 < \beta < 1$ is considered, which can not be handled by our method.

Finally, concerning the growth conditions given by (A) or (1.5), (1.6) respectively, it should be noted that for the case $h(u) = |u|^{\beta-1}u$, $\beta > 1$ KUSANO and NAITO proved that if

$$\frac{rq'(r)}{q(r)} \geq -\frac{N+2-\beta(N-2)}{2} \qquad r > 0,$$

then any classical radial solution of (1.3) does not change sign (cf. [14], [15]). Moreover they show in that if

$$\frac{rq'(r)}{q(r)} < -\frac{N+2-\beta(N-2)}{2} \qquad r > 0,$$

any classical radial solution of (1.3) has an infinite number of nodal domains.

In the nonradial case we were not able to establish the existence of solutions with n nodal domains for given $n \geq 3$, and it is the authors belief that in general this is impossible. Note nevertheless that TSHINANGA [23] established the existence of infinitly many solutions (without nodal characterization) under slightly different assumptions.

We prove our assertions developing a new variational approach which is related to methods introduced in [11], [10] for the treatment of *sublinear* equations. This approach is based on an examination of the geometry of the functional ψ around a Ljusternik-Schnirelman level by a 'comparison' with 'frozen' weighted linear eigenvalue problems of the form

$$-\Delta v = \lambda f(x, u^2)v \qquad\qquad v \in D^{1,2}(\mathbb{R}^N). \tag{1.8}$$

More precisely, we will detect nodal solutions of (1.1) by solving (1.8) with the additional information that u equals v, and that $\lambda = 1$ satisfies a given (linear) minimax characterization. This enables us to carry over nodal characterizations which are known for linear problems (e.g. problems of Sturm-Liouville type) to the nonlinear situation. Furthermore we feel that it leads to a deeper understanding of the relationship between linear and nonlinear minimax principles.

Finally, Corollary 1.4 seems to be a special case of a more general principle that might hold for many types of nonlinear problems, and which roughly says that the nondegeneracy of the eigenvalues of a perturbative family of linear operators implies the nondegeneracy of Ljusternik-Schnirelman values. For another type of equations we already proved an even stronger version dealing also with finite multiplicity (see [11]).

Actually our method strongly relies on the nondegeneracy of the eigenvalues of linear perturbed problems, and it is a difficult question how this nondegenracy might be ensured also in the *nonradial* case (Note that the linerization of (1.1) is an operator with purely continuous spectrum, so we can not even use perturbation expansions). Only the first eigenvalue is easily detected to be nondegenerate, and this is the reason for Theorem 1.2 to hold.

The paper is organized as follows: In section 2 we develop a variational comparison method in an abstract operator theoretical framework. This is motivated by the fact that for every $u \in D^{1,2} := D^{1,2}(\mathbb{R}^N)$ the multiplication operator $B(u) : D^{1,2} \to (D^{1,2})^*$ given by $v \mapsto f(\cdot, u^2(\cdot))v$ is compact ($(D^{1,2})^*$ denoting the topological dual of $D^{1,2}$), and that $v \in D^{1,2}$ is a weak solution of (1.8) if and only if it is an eigenfunction of the eigenvalue problem

$$J^{-1}B(u)v = \sigma v \tag{1.9}$$

with $\sigma = \frac{1}{\lambda}$ (Here $J : D^{1,2} \to (D^{1,2})^*$ denotes the canonical isometric isomorphism). Hence we are concerned with a family of operators of the form $J^{-1}B(u)$ on an abstract Hilbert space, and we derive existence results for an associated nonlinear eigenvalue problem (see Definition 2.2).

In section 3 we apply these results to the radially symmetric Emden-Fowler equation, and we prove Theorem 1.1 and Theorem 1.3(b). In section 4 we prove Theorem 1.2 and Theorem 1.3(a), considering now the nonsymmetric case.

Finally the paper contains a rather extensive appendix. In the first part we prove a compact embedding of the subspace of radial functions in $D^{1,2}$ into a space of continuous functions on the half-line, and in the second part we establish a uniqueness result for solutions of (1.8). Finally we prove an analogue of Courant's nodal domain theorem for eigenfunctions of (1.9).

Last but not least we remark that our method works even more easily for equations of the form

$$(-\Delta + q)u - f(x, u^2)u = \lambda u \qquad u \in H^{1,2}(\mathbb{R}^N), \tag{1.10}$$

assuming that $\lambda < \inf \sigma_{ess}(-\Delta + q)$ and that f induces compact multiplication operators $H^{1,2} \to H^{-1,2}$ given by $v \mapsto f(x, u^2)v$ for $u \in H^{1,2}$. There has been a lot of work on the existence of radial nodal solutions to (1.10), see e.g [12],[3] and [7], and we remark that CONTI *et al.* treated nonlinearities in [7] which are neither odd nor uniformly superlinear. On the other hand, in all of the above references, nonlinearities with supercritical behaviour in u or a singularity at the origin are not admitted. Moreover, the analogues of Theorem 1.2, Theorem 1.3 and Corollary (1.4) for (1.10) are new. Sometimes estimates on the number of nodal domains are derived with morse theory in case that the functional is C^2 (cf. e.g. [4],[5]), but this theory does not apply if f is just a Caratheodory function. For a more detailed consideration of (1.10) with the help of our methods, see [24].

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2 A variational comparison method for saddle points

Let X denote a real Hilbert space with scalar product $(\cdot|\cdot)$ and norm $\|\cdot\|$, denote X^* its topological dual and $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{R}$ the dual pairing. Furthermore consider a (nonlinear) map $B : X \to \mathcal{L}(X, X^*)$, which is supposed to satisfy the fundamental hypotheses:

(H1) B is continuous.

(H2) $B(u) \in \mathcal{L}(X, X^*)$ is a compact operator for every $u \in X$.

(H3) $B(0) = 0$, and $B(-u) = B(u) \quad$ for all $u \in X$.

(H4) $\langle B(u)v, w \rangle = \langle B(u)w, v \rangle$ for all $u, v, w \in X$.

(CC) ('Comparison Conditions') There is an even map $\varphi : X \to \mathbb{R}$, $\eta > 2$ such that for arbitrary vectors $u, v \in X$ there holds

  (i) $2(\varphi(v) - \varphi(u)) \geq \langle B(u)v, v \rangle - \langle B(u)u, u \rangle$
  (ii) The function $t \mapsto \langle B(tu)v, v \rangle$ is nondecreasing and increases strictly once it takes positive values.
  (iii) $0 \leq \eta\varphi(u) \leq \langle B(u)u, u \rangle$
  (iv) If $\langle B(u)v, v \rangle > 0$, then $\langle B(tv)v, v \rangle > 0$ for some $t > 0$.

We state some consequences of these assumptions. First note that

$$\varphi(0) = 0 \tag{2.1}$$

by (CC)(iii) and (H3). Moreover, (CC)(ii) and (H3) imply

$$\langle B(u)v, v \rangle \geq 0 \qquad (u, v \in X). \tag{2.2}$$

As a further consequence we observe

Lemma 2.1 *(i) There holds $\varphi \in C^1(X)$ with derivative given by*

$$\varphi'(u) = B(u)u \qquad (u \in X). \tag{2.3}$$

(ii) For every $u \in X$, $t \geq 1$ we have $\varphi(tu) \geq t^\eta \varphi(u)$.

Proof. (i) Since the map $u \mapsto B(u)u$ is continuous by (H1), it suffices to show that φ is Gateaux differentiable with Gateaux derivative given by (2.3). Therefore consider $u, v \in X$ and $t > 0$. Then, using (CC) and (H3),

$$\begin{aligned} 2\frac{\varphi(u+tv)-\varphi(u)}{t} &\geq \frac{1}{t}[\langle B(u)(u+tv),(u+tv)\rangle - \langle B(u)u,u\rangle] \\ &= 2\langle B(u)u,v\rangle + t\langle B(u)v,v\rangle \end{aligned}$$

as well as

$$\begin{aligned} 2\frac{\varphi(u+tv)-\varphi(u)}{t} &\leq \frac{1}{t}[\langle B(u+tv)(u+tv),(u+tv)\rangle - \langle B(u+tv)u,u\rangle] \\ &= 2\langle B(u+tv)u,v\rangle + t\langle B(u+tv)v,v\rangle. \end{aligned}$$

Passing to the limit $t \to 0$ we derive (2.3).
(ii) If $u \in X$ and $t \geq 1$, then

$$\begin{aligned} \log\varphi(tu) - \log\varphi(u) &= \int_1^t \frac{\langle B(su)su,u\rangle}{\varphi(su)}\,ds \\ &= \int_1^t \frac{\langle B(su)su,su\rangle}{s\varphi(su)}\,ds \\ &\geq \eta\int_1^t \frac{ds}{s}\,ds = \eta\log t \end{aligned}$$

by (CC)(iii), hence $\varphi(tu) \geq t^\eta\varphi(u)$.

In particular, Lemma 2.1(i) implies that φ is continuous. Moreover, in view of (CC)(ii) we have

$$\varphi(u) > 0 \iff \langle B(u)u,u\rangle > 0. \tag{2.4}$$

Now denote $J : X \to X^*$ the canonical isometric isomorphism. Since for every $u \in X$ the operator $J^{-1}B(u) : X \to X$ is compact, symmetric and nonnegative by (H2), (H4) and (2.2), its spectrum consists of a decreasing sequence of eigenvalues given by

$$\sigma_k(u) := \sup_{\substack{V \leq X \\ \dim V = k}} \inf_{v \in V} \frac{(J^{-1}B(u)v|v)}{(v|v)} = \sup_{\substack{V \leq X \\ \dim V = k}} \inf_{v \in V} \frac{\langle B(u)v,v\rangle}{(v|v)}. \tag{2.5}$$

We remark that, since B is continuous, the functions $\sigma_k(\cdot) : X \to \mathbb{R}$ are continuous as well. In the rest of the section we are concerned with the following special *nonlinear eigenvalue* problem:

Definition 2.2 *Let $n \in \mathbb{N}$. A vector $u \in X$ is called a solution of Problem $(NEP)_n$ if*

$$J^{-1}B(u)u = u \tag{2.6}$$

and $\sigma_n(u) = 1$.

To treat this problem, we first define a family of symmetric bilinear forms by

$$[v, w]_u := (v|w) - \langle B(u)v, w\rangle \qquad \forall\, u, v, w \in X$$

and consider

$$\mathcal{N} := \{u \in X | [u, u]_u = 0\} - \{0\}.$$

Note that $\mathcal{N}$ is a closed and symmetric subset of X, since B is continuous and $B(0) = 0$. Furthermore, $\mathcal{N}$ contains all nontrivial solutions of (2.6). Let $\psi : X \to \mathbb{R}$ be given by

$$\psi := \frac{1}{2}\|u\|^2 - \varphi(u) \qquad (u \in X).$$

By (CC)(iii) we see that for $u \in \mathcal{N}$ there holds

$$\psi(u) \geq \left(\frac{1}{2} - \frac{1}{\eta}\right)\|u\|^2, \tag{2.7}$$

in particular ψ is non-negative on $\mathcal{N}$. The aim is to set up minimax principles for ψ on $\mathcal{N}$. Therefore consider

$$c_n := \inf_{\substack{A\in\Sigma(\mathcal{N})\\ \gamma(A)\geq n}} \sup_{u\in A} \psi(u) = \sup_{\substack{A\in\Sigma(\mathcal{N})\\ \gamma^*(A)\leq n-1}} \inf_{u\in A} \psi(u) \tag{2.8}$$

where we denote $\Sigma(\mathcal{N})$ the system of closed and symmetric subsets of $\mathcal{N}$, γ the Krasnosel'skii genus and

$$\gamma^*(A) := \sup\{\gamma(B) \mid B \in \Sigma(\mathcal{N}) \text{ and } B \cap A = \emptyset\} \qquad (A \in \Sigma(\mathcal{N})).$$

For the definition and properties of the Krasnosel'skii genus see e.g. [22, p. 94]. The equality in (2.8) is a consequence of the observation that

$$c_n = \inf\{c \in \mathbb{R} |\ \gamma(\mathcal{N} \cap \psi^{-1}(]-\infty, c]) \geq n\} = \sup\{c \in \mathbb{R} |\ \gamma(\mathcal{N} \cap \psi^{-1}(]-\infty, c]) < n\}$$

However, so far we have not ensured that $\mathcal{N}$ is nonempty (Note that $B \equiv 0$ satisfies our assumptions), in particular there could hold $c_n = +\infty$ for all n. Therefore we introduce the following hypothesis:

(GK) $\gamma(\mathcal{N}) = \infty$ (i.e., there is no continuous and odd map from $\mathcal{N}$ to a finite dimensional sphere).

Note nevertheless that, if (GK) holds, $\mathcal{N}$ still may not be homeomorphic to the unit sphere in X by radial projection, and in our applications this indeed is the case. The following lemma gives a more detailed description on the geometry of $\mathcal{N}$.

Lemma 2.3 *If $V \subset X$ is a finite dimensional subspace such that for every $v \in V \setminus \{0\}$ there is a number $t > 0$ such that $\langle B(tv)v, v\rangle > 0$, then $V \cap \mathcal{N}$ is homeomorphic to the unit sphere $S \subset V$ by radial projection. In particular $\gamma(V \cap \mathcal{N}) = \dim V$.*

Proof. First observe that by (CC)(ii) the radial projection $V \cap \mathcal{N} \to S$ is injective. Moreover, by assumption and (2.4) we find for every $v \in V$ a positive number t such that $\varphi(w) > 0$ for $w = tv$, hence (CC)(i) and Lemma 2.1(ii) yield

$$\langle B(sw)sw, sw\rangle \geq 2s^{\eta}\varphi(w) \geq \|sw\|^2 \tag{2.9}$$

for $s = s(w) > 0$ large enough. In particular there is a unique $t_v > 0$ such that $t_v v \in \mathcal{N}$. Since S is compact and B is continuous, the set $\{t_v \,|v \in S\}$ is bounded. Hence $V \cap \mathcal{N}$ is compact and the radial projection is continuous and bijective considered as a map $V \cap \mathcal{N} \longrightarrow S$. Thus it is a homeomorphism.

In the following lemma we see how condition (CC) allows us to 'compare' the family of eigenvalues $\sigma_n(u)$ of the operators $J^{-1}B(u)$ to the 'nonlinear' minimax value c_n.

Proposition 2.4 *Put $d := \inf_{u \in \mathcal{N}} \|u\| > 0$ and consider $u \in \mathcal{N}$. If $\sigma_n(u) \geq 1$, then there holds*

$$\psi(u) - c_n \geq \frac{d^2}{2}(\sigma_n(u) - 1). \tag{2.10}$$

Proof. Since $u \in \mathcal{N}$, condition (CC) implies

$$2(\psi(v) - \psi(u)) \leq [v, v]_u - [u, u]_u = [v, v]_u \tag{2.11}$$

for every $v \in X$. Now choose pairwise orthogonal eigenvectors $v_1, \ldots, v_n \in X$ corresponding to $J^{-1}B(u)$ and the eigenvalues $\sigma_1(u), \ldots, \sigma_n(u)$, and denote V the span of $v_1, ..., v_n$. Then clearly $\langle B(u)v, v\rangle > 0$ for $v \in V \setminus \{0\}$. By (CC)(iv) and Lemma 2.3 we conclude $\gamma(V \cap \mathcal{N}) = n$, and therefore

$$\begin{aligned} 2(c_n - \psi(u)) &\leq 2(\sup_{v \in \mathcal{N} \cap V} \psi(v) - \psi(u)) \\ &\leq \sup_{v \in \mathcal{N} \cap V} [v, v]_u \\ &\leq \sup_{v \in \mathcal{N} \cap V} \|v\|^2 (1 - \sigma_n(u)) \\ &\leq d^2(1 - \sigma_n(u)). \end{aligned} \tag{2.12}$$

Corollary 2.5 *Suppose $u \in \mathcal{N}$. Then $\psi(u) \leq c_n$ (resp. '$<$') implies $\sigma_n(u) \leq 1$ (resp. '$<$').*

This is an interesting observation especially for solutions of (2.6) on the ψ-level c_n, but it does not answer the question whether such solutions exist or not. Moreover, as stated in problem $(NEP)_n$, we intend to establish the stronger assertion $\sigma_n(u) = 1$ for such solutions. As a step in this direction, we look for a maximizing set related to the second characterization in (2.8), i.e. a closed and symmetric set $K \subset \mathcal{N}$ satisfying $\gamma^*(K) \leq n-1$ and $\inf_{u \in K} \psi(u) = c_n$. Therefore we fix $n \in \mathbb{N}$ for the rest of the section and denote $Q(u)$ the spectral projection associated with the operator $J^{-1}B(u)$ and the interval $[0, \sigma_n(u)]$. Moreover we define the *fixed point set*

$$K := \{u \in \mathcal{N} \mid Q(u)u = u\}.$$

Indeed we observe the following:

Lemma 2.6 *The set K is closed.*

Proof. Consider a sequence $(u_k)_k \subset K$ such that $u_k \to u$ in $\mathcal{N}$. Pick $j \in \mathbb{N}$ minimal such that $\sigma_j(u) = \sigma_n(u)$. If $j = 1$, then clearly $u \in K$. Hence consider the case $1 < j \leq n$, and denote $P(u_k)$ the spectral projection associated with the operator $J^{-1}B(u_k)$ and the eigenvalues $\sigma_1(u_k), ..., \sigma_{j-1}(u_k)$, $k \in \mathbb{N}$. Now choose a closed smooth path $\Gamma \subset \mathbb{C}$ surrounding $\sigma_1(u), ..., \sigma_{j-1}(u)$ but no other value $\sigma_i(u)$, $i \geq j$. Since the functions $\sigma_k(\cdot)$ are continuous, there is a number k_0 such that for $k \geq k_0$ the projection $P(u_k)$ is given by

$$P(u_k) := \frac{1}{2\pi i} \int_\Gamma (\lambda I - J(u_k))^{-1} d\lambda, \tag{2.13}$$

where we denoted $J(u_k) := J^{-1}B(u_k)$ (Strictly speaking, this holds true for the complexification of $P(u_k)$). From the relation

$$\begin{aligned} & (\lambda_1 I - J(u_k))^{-1} - (\lambda_2 I - J(u))^{-1} \\ = \; & (\lambda_2 I - J(u))^{-1}[(\lambda_2 - \lambda_1)I - (B(u) - B(u_k))](\lambda_1 I - J(u_k))^{-1} \qquad (\lambda_1, \lambda_2 \in \Gamma) \end{aligned}$$

we deduce that $(\lambda I - J(u_k))^{-1} \to (\lambda I - J(u))^{-1}$ in $\mathcal{L}(X)$ *uniformly* in $\lambda \in \Gamma$. Hence $P(u_k) \to P(u)$ in $\mathcal{L}(X)$ by (2.13), and therefore $P(u)u = \lim_{k \in \mathbb{N}} P(u_k)u_k = 0$, since $u_k \in K$ for all k. Recalling that $\sigma_n(u) = \sigma_j(u)$ we infer that $u \in K$, hence K is closed.

Lemma 2.7 *For $u \in K$ there holds:*
(a) $\psi(u) \geq c_n$
(b) If $\psi(u) = c_n$, then u is a solution of problem $(NEP)_n$.

Proof. Note that $u \in K$ implies $\sigma_n(u) \geq 1$, therefore (a) follows directly from proposition 2.4. Moreover, if in addition $\psi(u) = c_n$, then (2.10) implies $\sigma_n(u) = 1$, and from $(J^{-1}B(u)u|u) = (u|u)$ and $u \in \mathcal{R}(Q(u))$ we conclude $J^{-1}B(u)u = u$. Thus u is a solution of $(NEP)_n$.

As a consequence, we might find solutions of $(NEP)_n$ as minimizers u for $\psi|_K$, provided that K is nonempty and that

$$c_n = \inf_{u \in K} \psi(u) \tag{2.14}$$

In view of (2.8) and Lemma 2.7(a) this holds under the assumption

(EC) $\gamma^*(K) \leq n-1$,

which together with (GK) in particular implies, in particular, that K is nonempty. At this point it might not be clear that this condition can be satisfied, but it actually turns out that we can ensure it in our applications.

At any rate, if (EC) is satisfied, (2.14) holds and Lemma 2.7(b) shows that every minimizer for ψ in K is a solution of $(NEP)_n$. To establish the existence of minimizers, we examine minimizing sequences in K, and we need the following estimate:

Lemma 2.8 *There holds*

$$\|Ju - B(u)u\|_{X^*} \leq \|u\| \left[(\sigma_n(u) - 1) + [(\sigma_n(u) - 1)\sigma_n(u)]^{\frac{1}{2}} \right]$$

for every $u \in K$.

We postpone the proof of this assertion and instead we first use it to prove the following:

Theorem 2.9 *Assume that (GK) and (EC) are satisfied, and that $(u_j) \subset K$ is a minimizing sequence for ψ in K. Then (u_j) is bounded and $\|Ju_j - B(u_j)u_j\|_{X^*} \to 0$ for $j \to \infty$.*

Proof. By (2.7) we observe that $(u_j)_j$ is bounded. Moreover (2.10) yields

$$o(1) = \psi(u) - c_n \geq \frac{d^2}{2}(\sigma_n(u) - 1) \geq 0,$$

hence $\sigma_n(u_j) \to 1$. The assertion now follows from Lemma 2.8.

Proof of Lemma 2.8. Fix $u \in K$. Then clearly $\sigma_n(u) \geq 1$, and $\sigma_n(u)(\cdot|\cdot) - (J^{-1}B(u) \cdot |\cdot)$ defines a (semidefinite) scalar product on $\mathcal{R}(Q(u)) \subset X$. Set $w := u - J^{-1}B(u)u \in \mathcal{R}(Q(u))$, then by the corresponding *Cauchy-Schwartz- Inequality*

$$\begin{aligned}
\sigma_n(u)(u|w) - (J^{-1}B(u)u|w) &\leq [\sigma_n(u)\|u\|^2 - (J^{-1}B(u)u|u)]^{\frac{1}{2}} \times \\
&\quad\; [\sigma_n(u)\|w\|^2 - (J^{-1}B(u)w|w)]^{\frac{1}{2}} \\
&\leq [(\sigma_n(u) - 1)\|u\|^2]^{\frac{1}{2}} [\sigma_n(u)\|w\|^2]^{\frac{1}{2}} \\
&\leq \|u\|[(\sigma_n(u) - 1)\sigma_n(u)]^{\frac{1}{2}} \|w\|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|w\|^2 &= (u|w) - (J^{-1}B(u)u|w) \\
&\leq \sigma_n(u)(u|w) - (J^{-1}B(u)u|w) + |1 - \sigma_n(u)|(u|w) \\
&\leq \|u\|[(\sigma_n(u) - 1)\sigma_n(u)]^{\frac{1}{2}} \|w\| + (\sigma_n(u) - 1)\|u\|\|w\|,
\end{aligned}$$

hence

$$\|Ju - B(u)u\|_{X^*} = \|w\| \leq \|u\| \left[(\sigma_n(u) - 1) + [(\sigma_n(u) - 1)\sigma_n(u)]^{\frac{1}{2}} \right].$$

Corollary 2.10 *Assume that (GK) and (EC) are satisfied, and that the (nonlinear) operator $B : X \to \mathcal{L}(X, X^*)$ is compact. Then ψ attains its minimum on K, hence there is a solution of $(NEP)_n$.*

Proof. Since B is compact and (H2) holds, the operator $\tilde{B} : X \to X^*$ defined by $\tilde{B}(u) := B(u)u$ is compact as well. Now if $(u_j)_j$ is a minimizing sequence for ψ in K, then (u_j) is bounded by (2.7) and there is $w \in X^*$ such that $B(u_j)u_j \to w$ after passing to a subsequence. By Theorem 2.9 we conclude that $u_j \to J^{-1}(w)$, and $\psi|_K$ attains its minimum at $J^{-1}(w)$.

3 The radial case

In this section we always assume that f can be written in the form $f(x, u^2) \equiv \mathfrak{f}(|x|, u^2)$, and that $(\mathcal{F}_1)$, $(\mathcal{F}_2)$, $(\mathcal{F}_3)$ and (A) are satisfied. First we adapt the notation of section 2 to this situation. Thus let X denote the Hilbert space of radially symmetric functions in $D^{1,2}(\mathbb{R}^N)$. We note that every $u \in X$ is continuous on $\mathbb{R}^N \setminus \{0\}$ and there holds

$$|u(r)| \le C\|u\| r^{\frac{2-N}{2}} \tag{3.1}$$

with a constant $C = C(N) > 0$ (see e.g. [16, p.55]). Furthermore we observe that in the distributional sense $-\Delta$ coincides with the canonical isometric isomorphism $J : X \to X^*$ on X.

Lemma 3.1

a) There is a completely continuous, i.e. compact and continuous map $B : X \to \mathcal{L}(X, X^*)$ defined by

$$\langle B(u)v, w\rangle := \int_{\mathbb{R}^N} \mathfrak{f}(|x|, u^2)v(x)w(x)dx$$

such that $B(u)$ is a compact linear operator for each $u \in X$.

b) For every $u \in X$ there exists

$$\varphi(u) := \frac{1}{2}\int_{\mathbb{R}^N}\int_0^{u^2(x)} \mathfrak{f}(|x|, s)\, ds\, dx \qquad (u \in X). \tag{3.2}$$

Moreover, the functional $\varphi : X \to \mathbb{R}$ and the map B satisfy condition (CC) from section 2.

We postpone the (somewhat technical) proof for a moment and infer that, as a consequence, the map B satisfies (H1)-(H4) as well as (CC). Hence we may consider problem $(NEP)_n$ (as stated in section 2), and we observe that every weak solution $u \in X$ of (1.1) solves (2.6) as well. Moreover there holds the following:

Proposition 3.2 *If $u \in X$ is a solution of $(NEP)_n$, then $u \in C^1(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N)$ for all $1 \le p < \infty$, and u is a solution of (1.1) having precisely n nodal surfaces.*

Hence Theorem 1.1 will follow from Corollary 2.10 as soon as we have ensured the remaining assumptions. We will prove Proposition 3.2 at the end of this section, and we first turn to the

Proof of Lemma 3.1. In view of (I3) we find $\mu > \dfrac{N-2}{2}$, $\nu < \dfrac{N-2}{2}$ such that

$$\delta_1 := \alpha_1 + N - 2\mu(\beta_1 + 1) > 0 \tag{3.3}$$

and

$$\delta_2 := \alpha_2 + N - 2\nu(\beta_2 + 1) < 0. \tag{3.4}$$

Fix such values μ and ν and note that by Lemma 5.1 of the appendix there is a compact embedding $i : X \to C_{\mu,\nu}$. Now we turn to the proof of a):

Consider the following factorization for B:

$$X \overset{i}{\hookrightarrow} C_{\mu,\nu} \overset{b}{\to} \mathcal{L}(C_{\mu,\nu}, C_{\mu,\nu}^*) \overset{j}{\to} \mathcal{L}(X, X^*).$$

Here the map b is defined by

$$\langle b(u)v, w\rangle := |S^{N-1}| \int_0^\infty r^{N-1}\mathfrak{f}(r, u^2)v(r)w(r)dr, \tag{3.5}$$

($|S^{N-1}|$ denoting the volume of the unit sphere in $\mathbb{R}^N$), and j maps a linear operator $D \in \mathcal{L}(C_{\mu,\nu}, C_{\mu,\nu}^*)$ to $i^*Di \in \mathcal{L}(X, X^*)$. We show that b is well-defined and continuous: First consider a fixed $u \in C_{\mu,\nu}$. Then for all $v, w \in C_{\mu,\nu}$ there holds

$$|\mathfrak{f}(r, u^2)v(r)w(r)| \quad \le \quad \begin{cases} c_1 r^{\delta_1 - N}\|u\|_{\mu,\nu}^{2\beta_1}\|v\|_{\mu,\nu}\|w\|_{\mu,\nu} & \text{for} \quad 0 < r \le 1 \\ c_2 r^{\delta_2 - N}\|u\|_{\mu,\nu}^{2\beta_2}\|v\|_{\mu,\nu}\|w\|_{\mu,\nu} & \text{for} \quad 1 < r < \infty. \end{cases}$$

By integration we conclude that $b(u)$ is a well defined element of $\mathcal{L}(C_{\mu,\nu}, C_{\mu,\nu}^*)$. In order to show the continuity of b, pick a sequence $(u_n)_n \subset C_{\mu,\nu}$ with $u_n \to u \in C_{\mu,\nu}$. Given $\varepsilon > 0$, choose $0 < r_1 < 1$ and $1 < r_2 < \infty$ in such a way that

$$2c_1 M^{2\beta_1}\frac{r_1^{\delta_1}}{\delta_1} < \varepsilon \tag{3.6}$$

and

$$-2c_2 M^{2\beta_2}\frac{r_2^{\delta_2}}{\delta_2} < \varepsilon, \tag{3.7}$$

where $M := \sup\limits_{n\in\mathbb{N}}\{\|u_n\|_{\mu,\nu}\}$. Since $\mathfrak{f}(r, u_n^2(r))$ converges uniformly to $\mathfrak{f}(r, u^2(r))$ on the compact subinterval $[r_1, r_2]$, there is an $n_0 \in \mathbb{N}$ such that

$$\int_{r_1}^{r_2} \max\{r^{N-1-2\mu}, r^{N-1-2\nu}\}|\mathfrak{f}(r, u^2(r)) - \mathfrak{f}(r, u_n^2(r))|\, dr < \varepsilon$$

for $n \ge n_0$. But, using (3.6), we also have

$$\begin{aligned} \int_0^{r_1} r^{N-1-2\mu}|\mathfrak{f}(r, u^2(r)) - \mathfrak{f}(r, u_n^2(r))|\, dr \quad &\le \quad \int_0^{r_1} r^{N-1-2\mu}(|\mathfrak{f}(r, u^2(r))| + |\mathfrak{f}(r, u_n^2(r))|)\, dr \\ &\le \quad 2c_1 M^{2\beta_1}\int_0^{r_1} r^{\delta_1 - 1} \\ &\le \quad 2c_1 M^{2\beta_1}\frac{r_1^{\delta_1}}{\delta_1} \\ &< \quad \varepsilon. \end{aligned}$$

Similarly (3.7) implies

$$\int_{r_2}^{\infty} r^{N-1-2\nu}|\mathfrak{f}(r,u^2(r)) - \mathfrak{f}(r,u_n^2(r))|\,dr < \varepsilon. \tag{3.8}$$

Combining these estimates, we obtain for $n \geq n_0$

$$\begin{aligned}\langle (b(u)-b(u_n))v,w\rangle &= |S^{N-1}|\int_0^{\infty} r^{N-1}\mathfrak{f}(r,u^2)v(r)w(r)dr \\ &\leq 3\varepsilon|S^{N-1}|\|v\|_{\mu,\nu}\|w\|_{\mu,\nu},\end{aligned}$$

and this shows the continuity of b.
Considering now the factorization given above, we see that indeed the map B is completely continuous. Furthermore the range of B consists only of *compact* operators, since the same holds true for the map j. Thus a) is proved.
b): For $u \in X$, $0 < r \leq 1$ we have

$$\begin{aligned}\int_0^{u^2(x)} \mathfrak{f}(r,s)\,ds &\leq c_1 r^{\alpha_1}\int_0^{u^2(x)} s^{\beta_1}\,ds \\ &\leq \frac{c_1}{\beta_1+1}r^{\alpha_1}|u(x)|^{2(\beta_1+1)} \\ &= \frac{c_1}{\beta_1+1}\|u\|_{\mu,\nu}^{2(\beta_1+1)}r^{\delta_1-N}\end{aligned}$$

Analoguesly we derive for $1 \leq r < \infty$

$$\int_0^{u^2(x)} \mathfrak{f}(r,s)\,ds \leq \frac{c_2}{\beta_2+1}\|u\|_{\mu,\nu}^{2(\beta_2+1)}r^{\delta_2-N}$$

Therefore the integral on the right hand side of (3.2) exists.
To prove (CC)(i)-(iv), note that for arbitrary $r \in]0,\infty]$ the function

$$F(r,\cdot) := \int_0^{(\cdot)} \mathfrak{f}(r,s)ds$$

is convex on $[0,\infty[$, since f satisfies $(\mathcal{F}_1)$. In particular, for $u,v \in X$ and $x \in \mathbb{R}^N \setminus \{0\}$ there holds

$$\begin{aligned}F(|x|,v^2(x)) - F(|x|,u^2(x)) &\geq \partial_2 F(|x|,u^2(x))(v^2(x)-u^2(x)) \\ &= \mathfrak{f}(|x|,u^2(x))(v^2(x)-u^2(x)).\end{aligned}$$

Noting that

$$2\varphi(u) = \int_{\mathbb{R}^N} F(|x|,u^2(x))\,dx, \qquad (u \in X)$$

condition (CC)(i) follows by integration. Now (CC)(ii) and (CC)(iii) are immediate consequences of $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$. To ensure (CC)(iv), note that $\langle B(u)v,v\rangle > 0$ implies that $I(f) \cap \operatorname{supp}(v)$ is a set of positive measure, where

$$I(f) := \{r \in]0,\infty[\ |\ \exists s > 0 \text{ s.t. } \mathfrak{f}(r,s) > 0\}, \tag{3.9}$$

hence $\langle B(tv)v,v\rangle > 0$ for $t > 0$ large enough.

Lemma 3.3 *Condition (GK) is satisfied, i.e.* $\gamma(\mathcal{N}) = \infty$.

Proof. We prove this by constructing for given $m \in \mathbb{N}$ an m-dimensional subspace $V_m \subset X$ such that for every $v \in V_m \setminus \{0\}$ there exists $t > 0$ such that

$$\langle B(tv)v, v\rangle > 0, \tag{3.10}$$

which implies $\gamma(\mathcal{N} \cap V_m) = m$ by Lemma 2.3. Therefore choose m linearly independent functions $w_i : [0,\infty[\to \mathbb{R}$ which are real analytic on $]0,\infty[$, rapidly decreasing at infinity and satisfy $w_i'(0) = 0$ (e.g. one can take linear combinations of Hermite functions). Define $v_i \in X$ by $v_i(x) := w_i(|x|)$ and define V_m as the span of the v_i. Since $I(f)$ (as defined in (3.9)) is a set of positive measure by $(\mathcal{F}_3)$, any $v \in V_m \setminus \{0\}$ cannot vanish on $I(f)$, and we conclude that there is a number $t > 0$ such that (3.10) holds.

As the next step we want to ensure condition (EC) for given $n \in \mathbb{N}$, and we therefore again make use of the radial structure of the problem. Observe that (EC) holds if there are $n-1$ continuous and odd functions $g_i : \mathcal{N} \to \mathbb{R}$ such that

$$g_i(u) = 0 \;\forall\, i \quad \Longrightarrow u \in K. \tag{3.11}$$

Now the idea is the following: For $i = 1, ..., n-1$, $u \in X$ define $g_i(u) = h(0)$, where $h := P_i(u)u$, and $P_i(u) \in \mathcal{L}(X)$ is the spectral projection associated with the eigenvalue $\sigma_i(u)$ of the operator $J^{-1}B(u)$. Clearly g_i is an odd function. Moreover we observe:

Lemma 3.4 *For* $u \in \mathcal{N}$ *and all* $i \in \mathbb{N}$ *there holds* $\sigma_i(u) > 0$, *and* $\sigma_i(u)$ *is nondegenerate. Furthermore the map* $u \mapsto P_i(u)$ *is continuous.*

Proof. We recall the variational characterization

$$\sigma_i(u) := \sup_{\substack{V \subset X \\ \dim V = i}} \inf_{v \in V} \frac{\langle B(u)v, v\rangle}{(v|v)} \tag{3.12}$$

for each $i \in \mathbb{N}$. Now fix $u \in \mathcal{N}$. Then the measure of the set $\{x \in \mathbb{R}^N \mid \mathfrak{f}(|x|, u^2(x)) > 0\}$ is positive, and we may conclude $\sigma_i(u) > 0$ by testing with an i-dimensional subspace $V_i \subset X$ constructed as in the proof of Lemma 3.3. If u_i is a corresponding eigenfunction, there holds

$$-\Delta u_i = \frac{1}{\sigma_i(u)} \mathfrak{f}(r, u^2) u_i$$

at least in distributional sense. We first show

$(*) \qquad u_i \in C(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{0\}) \cap W^{2,p}_{loc}(\mathbb{R}^N) \quad$ for some $p > \frac{N}{2}$

Since the function $\mathfrak{f}(\cdot, u^2(\cdot))u_i(\cdot)$ is locally bounded on $\mathbb{R}^N \setminus \{0\}$ (recall that u is continuous on $\mathbb{R}^N \setminus \{0\}$), elliptic regularity yields $u_i \in W^{2,p}_{loc}(\mathbb{R}^N \setminus \{0\})$ for $1 \le p < \infty$. Hence $u_i \in C^1(\mathbb{R}^N \setminus \{0\})$ by Sobolev embeddings. Note furthermore that for every radially symmetric $\varphi \in C_0^\infty(B_1(0))$ we have

$$\begin{aligned} \|(\mathfrak{f}(\cdot, u^2(\cdot))^{\frac{1}{2}}\varphi\|^2_{L^2(\mathbb{R}^N)} &= \langle B(u)\varphi, \varphi\rangle \\ &\le \|B(u)\|\|\varphi\|^2 \\ &\le C\|B(u)\|\|(1-\Delta)^{\frac{1}{2}}\varphi\|_{L^2(\mathbb{R}^N)} \end{aligned} \tag{3.13}$$

with a constant $C > 0$ (Here $(1-\Delta)^{\frac{1}{2}}$ is understood as a pseudodifferential operator). Since the operators $-\Delta$ and $\mathfrak{f}(\cdot, u^2(\cdot))$ conserve radial symmetry, (3.13) holds even for arbitrary $\varphi \in C_0^\infty(B_1(0))$. Therefore [1, Theorem 5.1] implies that $u_i \in L^\infty(B_{1/2}(0))$. Now (A) yields $\mathfrak{f}(\cdot, u_i^2(\cdot))u_i \in L^p((B_{1/2}(0))$ for some $p > \frac{N}{2}$, hence $u_i \in W^{2,p}(B_{1/2}(0))$ by elliptic regularity. Finally $u_i \in C(B_{1/2}(0))$ by Sobolev embeddings, hence $(*)$ holds.
To conclude that $J^{-1}B(u)$ has only simple eigenvalues we apply Lemma 5.2, noting that in view of (3.1) the function $g(r) := \mathfrak{f}(r, u^2(r))$ satisfies $0 \le g(r) \le \text{const}\, r^\alpha$ for $r \in]0,1]$ with $\alpha := \alpha_1 - (N-2)\beta_1 > -2$.
Using the nondegeneracy of the values $\sigma_i(u), u \in \mathcal{N}$, the continuity of P_i now follows by a similar argument as in the proof of Lemma 2.6.

We conclude by the preceding Lemma that the maps $h_i : \mathcal{N} \to X$, $h_i(u) := P_i(u)u$ are continuous. Now fix $i \in \mathbb{N}$. To show the continuity of g_i, pick a sequence $(u_n) \subset \mathcal{N}$ such that $u_n \to u \in \mathcal{N}$, and denote $M := \{u_n, u\}$. Thus M is compact, therefore we conclude by (3.13) that

$$\|(\mathfrak{f}(\cdot, u^2(\cdot))^{\frac{1}{2}}\varphi\|^2_{L^2(\mathbb{R}^N)} \le C\|B(u)\|\|(1-\Delta)^{\frac{1}{2}}\varphi\|_{L^2(\mathbb{R}^N)} \le C_1\|(1-\Delta)^{\frac{1}{2}}\varphi\|_{L^2(\mathbb{R}^N)}$$

for all $u \in M$ and $\varphi \in C_0^\infty(B_1(0))$. A closer look at [1, Theorem 5.1] tells us that $C_2 := \sup\limits_{u\in M} \|h_i(u)\|_{L^\infty(B_{1/2}(0))} < \infty$. Moreover, in view of (A) there is a number $p > \frac{n}{2}$ such that $q := p(\alpha_1 - (N-2)\beta_1) > -N$, hence there holds

$$\begin{aligned}
\int_{|x|\le\frac{1}{2}} |\Delta h_i(u)|^p &= \frac{1}{\sigma_i(u)} \int_{|x|\le\frac{1}{2}} [\mathfrak{f}(|x|, u^2(|x|))(h_i(u))(|x|)]^p \, dx \\
&\le \frac{1}{\sigma_i(u)} c_1 C_2^p \int_0^{\frac{1}{2}} r^{N-1}[r^{\alpha_1}|u(r)|^{2\beta_1}]^p dr \\
&= C_3\|u\|^{2\beta_1 p} \int_0^1 r^{N-1-q}\, dr \\
&\le C_4
\end{aligned}$$

for $u \in M$ (again we used (3.1) and the fact that $\sigma_i(u_n) \to \sigma_i(u) > 0$). Therefore the set $h_i(M)$ is bounded in $W^{2,p}(B_{\frac{1}{2}}(0))$ due to the Calderon-Zygmund inequality (see e.g. [9, p.230]), and by the Sobolev embedding theorem it is relatively compact in $C(B_{\frac{1}{2}}(0))$. Hence $h_i(M)$ is equicontinuous. Now since (3.1) implies $h_i(u_n) \to h_i(u)$ pointwise on $\mathbb{R}^N \setminus \{0\}$, we conclude $h_i(u_n)(0) \to h_i(u)(0)$, and this shows the continuity of g_i.
Finally Lemma 5.2 implies that if $g_i(u) = 0$, then $P_i(u)u = h_i(u) = 0$, hence the functions g_i satisfy (3.11). Therefore we have proved the following:

Lemma 3.5 *Condition (EC) is satisfied for every $n \in \mathbb{N}$.*

Now we can complete the proofs of the radial Theorems:

Proof of Theorem 1.1. Fix $n \in \mathbb{N}$. The conditions (H1)-(H4) and (CC) are satisfied by Lemma 3.1, whereas (GK) and (EC) hold by Lemma 3.3 and Lemma 3.5 respectively.

Finally, the operator $B : X \to \mathcal{L}(X, X^*)$ is compact by Lemma 3.1, hence by Corollary 2.10 there is a solution of $(NEP)_n$. Thus Theorem 1.1 follows from Proposition 3.2.

Proof of Theorem 1.3(b):. A weak radial solution u of (1.1) is an eigenfunction of the operator $J^{-1}\tilde{g} : X \to X$ corresponding to the eigenvalue $\sigma(u) = 1$, where $\tilde{g} : X \to X^*$ corresponds to $g := \mathfrak{f}(\cdot, u^2(\cdot))$ in the way described in Theorem 5.3. As in the proof of Lemma 3.4 we see that u is continuous and that g is locally bounded on $\mathbb{R}^N \setminus \{0\}$, hence the assertion follows from a combination of Theorem 5.3 (together with the remark following it) and Corollary 2.5.

We conclude the section with the

Proof of Proposition 3.2. By $(*)$ (see the proof of Lemma 3.4) there holds $u \in C(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{0\}) \cap W^{2,p}_{loc}(\mathbb{R}^N)$ for some $p > \frac{N}{2}$. Viewed as a function of $r = |x|$, this implies $u \in C^1(0, \infty)$, u' is absolutely continuous and that

$$(r^{N-1}u')' + r^{N-1}\mathfrak{f}(r, u^2)u = 0$$

for $r > 0$. Moreover, there are $n-1$ linearly independent functions $u_i \in C[0, \infty) \cap C^1(0, \infty)$ such that u_i' is absolutely continuous on $(0, \infty)$ and

$$(r^{N-1}u_i')' + \frac{1}{\sigma_i(u)} r^{N-1}\mathfrak{f}(r, u^2)u_i = 0.$$

By seperation theorems of Sturm type we conclude inductively that $u = u_n$ has at least n nodal domains. On the other hand there cannot be more by Theorem 5.3 and the remark following it. Hence the assertion is proved.

4 The nonradial case

In this section we always assume that $(\mathcal{F}_1)$, $(\mathcal{F}_2)$, $(\mathcal{F}_3)$ and (B) are satisfied. Now we apply the theory of section 2 to $X := D^{1,2}(\mathbb{R}^N)$, and again we observe that, in distributional sense, $-\Delta$ coincides with the canonical isometric isomorphism $J : X \to X^*$. As in the previous section we have

Lemma 4.1 *a) There is a continuous map $B : X \to \mathcal{L}(X, X^*)$ defined by*

$$\langle B(u)v, w\rangle := \int_{\mathbb{R}^N} f(x, u^2)v(x)w(x)dx,$$

and $B(u)$ is a compact linear operator for each $u \in X$.
b) For every $u \in X$ there exists

$$\varphi(u) := \frac{1}{2}\int_{\mathbb{R}^N}\int_0^{u^2(x)} f(x, s)\, ds\, dx \qquad (u \in X).$$

Moreover, the functional $\varphi : X \to \mathbb{R}$ and the map B satisfy (CC).

Results by SCHNEIDER [20] suggest that B is *completely continuous* as in the radial case, but we did not examine this in detail. Instead we will use an observation of TSHINANGA [23].

Proof of Lemma 4.1. a) Set $p := 2^* := \frac{2N}{N-2}$ and consider the following factorization for B:

$$X \overset{i}{\hookrightarrow} L^p(\mathbb{R}^N) \xrightarrow{f_*} L^{\frac{N}{2}}(\mathbb{R}^N) \xrightarrow{g} \mathcal{L}(L^p(\mathbb{R}^N), L^{p'}(\mathbb{R}^N)) \xrightarrow{h} \mathcal{L}(X, X^*)$$

Here i denotes the Sobolev embedding, and $(f_*(u))(x) := f(x, u^2(x))$. Moreover g is defined by $g(v)w := vw$ for $v \in L^{\frac{N}{2}}(\mathbb{R}^N)$, $w \in L^p(\mathbb{R}^N)$, while h maps a linear operator $S \in L^{\frac{N}{2}}(\mathbb{R}^N)$ to i^*Si (i^* denoting the dual of i). Obviously i, g and h are continuous linear Operators. To ensure the continuity of f_*, we only have to check that it is bounded (cf. [17, p.22]). Therefore note that for $u \in L^p(\mathbb{R}^N)$ we have by Hölder's Inequality:

$$\begin{aligned}
\int_{\mathbb{R}^N} |f(x,u^2)|^{\frac{N}{2}} &\leq C \int_{\mathbb{R}^N} \left[\frac{1}{(1+|x|)^a}|u|^{2\gamma}\right]^{\frac{N}{2}} \\
&\leq C \int_{\mathbb{R}^N} \frac{1}{(1+|x|)^{\frac{aN}{2}}}|u|^{\gamma N} \\
&\leq C\|u\|_p^{\gamma N}\left(\int_{\mathbb{R}^N}\left(\frac{1}{1+|x|}\right)^{\frac{\tau' aN}{2}}\right)^{\frac{1}{\tau'}}, \qquad (4.1)
\end{aligned}$$

where τ' is the conjugate exponent of $\tau := \frac{p}{N\gamma} = \frac{2}{(N-2)\gamma} > 1$. A short calculation shows $\frac{\tau' aN}{2} = \frac{aN}{2-(N-2)\gamma} > N$ in view of (B), hence f_* is a bounded operator, and therefore B is continuous. It remains to show that $B(u) \in \mathcal{L}(X, X^*)$ is compact for every $u \in X$. Since the compact linear operators form a closed subspace of $\mathcal{L}(X, X^*)$, it suffices to consider $u \in C_0^\infty$. Therefore suppose that u vanishes on $\mathbb{R}^N \setminus \Omega$, $\Omega \subset \mathbb{R}^N$ a bounded domain. In this case $B(u)$ factorizes in the form

$$X \overset{r}{\hookrightarrow} W^{1,2}(\Omega) \overset{j}{\hookrightarrow} L^2(\Omega) \xrightarrow{\rho_u} L^2(\Omega) \xrightarrow{(r\circ j)^*} X^*$$

where r denotes the canonical restriction, j the *compact* Sobolev embedding, and ρ_u is given by $\rho_u(v) := f(\cdot, u^2(\cdot))v$. Hence $B(u)$ is compact.
b) There holds

$$\int_0^{u^2(x)} f(x,s)\,ds \leq \frac{C}{(\gamma+1)(1+|x|)^a}|u(x)|^{2\gamma+2} \qquad (4.2)$$

where the right hand side of (4.2) is an L^1-function by (B). Hence φ is well defined. (CC) follows exactly as in the radial case.

As a consequence, B satisfies (H1)-(H4) as well as (CC). Now we are ready to prove our nonradial results.

Proof of Theorem 1.2. We ensure the assumptions of Theorem 2.9 in case that $n \in 1, 2$. Therefore observe that Condition (GK) can be proved just as in the radial case (cf. Lemma 3.3). We check (EC) for the sets K associated with $n = 1$ and $n = 2$ respectively: In case

$n = 1$ the set K equals $\mathcal{N}$, hence (EC) holds trivially. In case $n = 2$, note that by the same arguments as in Lemma 3.4 there holds $\sigma_1(u) > 0$ for every $u \in \mathcal{N}$. Furthermore, by Theorem 5.3, every eigenfunction of $J^{-1}B(u)$ corresponding to $\sigma_1(u)$ does not change sign, in particular $\sigma_1(u)$ has to be nondegenerate. As in the proof of Lemma 3.4 we conclude that the spectral projection $P_1(u) \in \mathcal{L}(X)$ onto the corresponding eigenspace depends continuously on $u \in \mathcal{N}$. Now pick an arbitrary a.e. positive function $v \in X$ and observe that the map $h : \mathcal{N} \to \mathbb{R}$ defined by

$$u \overset{h}{\longmapsto} \int_{\mathbb{R}^N} vP_1(u)u$$

is odd and continuous. Moreover,

$$h(u) = 0 \quad \Longleftrightarrow \quad u \in K,$$

hence $\gamma^*(K) \leq 1$, and (EC) is satisfied for $n = 2$. Now fix $n \in \{1, 2\}$ and note that all assumptions of Theorem 2.9 are satisfied. Hence we get a minimizing sequence $(u_j)_j$ of ψ in K such that $Ju_j - B(u_j)u_j \to 0$ in X^*. By an observation of TSHINANGA [23, Lemma 2.2.] such a sequence contains a convergent subsequence (Note that even though in [23] it is supposed that $f \in C(\mathbb{R}^N \times \mathbb{R})$, the proof also works if f is just a Caratheodory function). Hence ψ attaines its minimum on K and every minimizer is a solution of (4.3) by Lemma 2.7. Thus, the following proposition completes the proof of Theorem 1.2.

Proposition 4.2 *If $n \in \{1, 2\}$ and $u_n \in X$ satisfies*

$$J^{-1}B(u_n)u_n = u_n, \qquad \sigma_n(u) = 1, \tag{4.3}$$

then $u_n \in C^1(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N)$ for $1 \leq p < \infty$, and u_n is a solution of (1.1) having precisely n nodal domains.

Proof. The function $u = u_n$ solves the equation

$$-\Delta u = f(x, u^2)u$$

in distributional sense, where $f(\cdot, u^2(\cdot)) \in L^{\frac{N}{2}}(\mathbb{R}^N)$ by (4.1). Therefore $u \in L^p_{loc}(\mathbb{R}^N)$ for $p \in [1, \infty)$, as proved e.g. in [22, p. 244]. Now (B) implies $f(\cdot, u^2(\cdot)) \in L^p_{loc}(\mathbb{R}^N)$ for $p \in [1, \infty)$, hence $u \in W^{2,p}_{loc}(\mathbb{R}^N)$ by elliptic regularity. Since $W^{2,p}$ is embedded in $C^1(\mathbb{R}^N)$ for p large enough, we conclude $u \in C^1(\mathbb{R}^N)$. By Theorem 5.3 the function u_1 does not change sign. Moreover,

$$\begin{aligned} 0 &= (u_1|u_2) = (J^{-1}B(u_1)u_1|u_2) = \langle B(u_1)u_1, u_2\rangle \\ &= \int_{\mathbb{R}^N} f(x, {u_1}^2)u_1u_2\, dx, \end{aligned}$$

hence u_2 has to change sign somewhere. By Theorem 5.3, u_2 has exactly two nodal domains.

Proof of Theorem 1.3(a):. A weak solution of (1.1) is an eigenfunction of the operator $J^{-1}\tilde{g} : X \to X$ corresponding to the eigenvalue $\sigma(u) = 1$, where $\tilde{g} : X \to X^*$ corresponds to $g := f(\cdot, u^2(\cdot))$ in the way described in Theorem 5.3. As in the proof of Proposition 4.2 we see that $u \in C^1(\mathbb{R}^N)$, hence g is locally bounded on $\mathbb{R}^N$. Therefore the assertion follows by combining Theorem 5.3 and Corollary 2.5.

5 Appendix

5.1 A compact embedding

In this section we prove a helpful Lemma which we do not claim to be new, but we could not find a version directly applicable to our situation. Therefore we add the proof here for the convenience of the reader.

In the following let X denote the Hilbert space of the radially symmetric functions in $D^{1,2}(\mathbb{R}^N)$ as in section 3. We first introduce for arbitrary $\mu, \nu \in \mathbb{R}$ the Banach spaces

$$C_{\mu,\nu} := \{u \in C(0,\infty) | \|u\|_{\mu,\nu} < \infty\},$$

where

$$\|u\|_{\mu,\nu} := \max\{\sup_{0<r\leq 1} r^\mu |u(r)|, \sup_{1\leq r<\infty} r^\nu |u(r)|\}.$$

The completeness of $C_{\mu,\nu}$ is obvious, since the map

$$u(r) \mapsto \begin{cases} r^\mu \|u(r)\| & \text{for} \quad 0 < r \leq 1 \\ r^\nu \|u(r)\| & \text{for} \quad 1 \leq r < \infty \end{cases}$$

defines an isometric isomorphism between $C_{\mu,\nu}$ and $C_b(0,\infty)$, i.e. the Banach space of bounded and continuous functions on $(0,\infty)$. Now there holds:

Lemma 5.1 *If $N \geq 3$, then there is a compact embedding $X \hookrightarrow C_{\mu,\nu}$ for $\mu > \frac{N-2}{2}$ and $\nu > \frac{N-2}{2}$.*

Proof. First note that for arbitrary $\mu, \nu, \mu', \nu' \in \mathbb{R}$ there are isometric isomorphisms $C_{\mu,\nu} \to C_{\mu',\nu'}$ given by

$$u(r) \mapsto v(r) := \begin{cases} r^{\mu-\mu'} u(r) & \text{for} \quad 0 < r \leq 1 \\ r^{\nu-\nu'} u(r) & \text{for} \quad 1 \leq r < \infty \end{cases}$$

Moreover,

$$C_{\mu,\nu} \hookrightarrow C_{\mu',\nu'}, \tag{5.1}$$

provided that $\mu' \geq \mu$, $\nu' \leq \nu$. Now consider $\mu = \frac{N-2}{2} + \varepsilon$, $\nu = \frac{N-2}{2} - \varepsilon'$, with ε, $\varepsilon' > 0$. We claim that there is a compact embedding $i : X \hookrightarrow C_{\mu,\nu}$ which factorizes in the following way:

$$X \xrightarrow{T_1} X_1 \xrightarrow{j} C_{-1+\varepsilon,-1-\varepsilon'} \xrightarrow{T_2} C_{\mu,\nu} \tag{5.2}$$

Here we denote $X_1 := \{u \in C_{-1,-1} \cap W^{1,2}_{loc}(0,\infty) \mid \|u\|_* < \infty\}$, where

$$\|u\|_*^2 := \int_0^1 (u')^2 + \int_1^\infty r^{-1}(u')^2.$$

Furthermore we endow X_1 with the norm $\|u\|_{X_1} := \|u\|_{-1,-1} + \|u\|_*$. The map T_1 is defined by

$$u(r) \overset{T_1}{\mapsto} v(r) := r^{\frac{N}{2}} u(r) \quad \text{for} \quad r > 0$$

while T_2 is defined analoguesly with corresponding exponent $-\frac{N}{2}$. Thus we have already seen that T_2 is continuous, as well as the inclusion $j : X_1 \hookrightarrow C_{-1,-1} \hookrightarrow C_{-1+\varepsilon,-1-\varepsilon'}$. Indeed this is a factorization for i, and therefore the compactness of i is ensured if we check the following two assertions:

(1) T_1 is continuous.

(2) j is compact.

Ad (1): As a consequence of (3.1), X is embedded in $C_{\frac{N-2}{2},\frac{N-2}{2}}$, and by definition T_1 maps $C_{\frac{N-2}{2},\frac{N-2}{2}}$ isometrically on $C_{-1,-1}$. On the other hand, for $u \in X$ there holds

$$\begin{aligned}
\|Tu\|_*^2 &= \int_0^1 [(r^{\frac{N}{2}}u(r))']^2\,dr + \int_1^\infty r^{-1}[(r^{\frac{N}{2}}u(r))']^2\,dr \\
&= \int_0^1 [\frac{N}{2}r^{\frac{N-2}{2}}u(r) + r^{\frac{N}{2}}u'(r)]^2\,dr + \int_1^\infty r^{-1}[\frac{N}{2}r^{\frac{N-2}{2}}u(r) + r^{\frac{N}{2}}u'(r)]^2\,dr \\
&\le \frac{N^2}{2}\int_0^1 r^{N-2}u^2(r)\,dr + 2\int_0^1 r^N(u'(r))^2\,dr \\
&\quad + \frac{N^2}{2}\int_1^\infty r^{N-3}u^2(r)\,dr + 2\int_1^\infty r^{N-1}(u'(r))^2\,dr \\
&\le c_1\|u\|^2_{\frac{N-2}{2},\frac{N-2}{2}} + c_2\|u\|^2 \\
&\le c_3\|u\|^2
\end{aligned}$$

and this implies the continuity of T_1.
Ad (2): Let $\mathcal{M}$ be an arbitrary bounded subset of X_1. Then there is a number $c > 0$ such that $\|f\|_{-1,-1} \le c$ for all $f \in \mathcal{M}$. We will show that for given $1 > \delta > 0$ there exists a relative compact set $\mathcal{M}_\delta \subset C_{-1+\varepsilon,-1-\varepsilon'}$ such that for $\delta' := 4\delta^{\min(\varepsilon,\varepsilon')}c$ there holds

$$j(\mathcal{M}) \subset U_{\delta'}(\mathcal{M}_\delta). \tag{5.3}$$

This implies the relative compactness of $\mathcal{M}$, so that j is compact, as claimed.
So pick an arbitrary $\delta \in (0,1)$ and consider the compact interval $K := [\delta, \frac{1}{\delta}]$. Since the embedding $W^{1,2}(K) \hookrightarrow C(K)$ is compact , the set

$$\tilde{\mathcal{M}} := \{f|_K \,:\, f \in \mathcal{M}\} \subset C(K)$$

is relatively compact. We define a continuous map $\tau : C(K) \to C_{-1+\varepsilon,-1-\varepsilon'}$ by

$$\tau f(r) := \begin{cases} \frac{r}{\delta}f(\delta) & \text{for} \quad 0 < r \le \delta \\ f(r) & \text{for} \quad \delta \le r \le \frac{1}{\delta} \\ r\delta f(\frac{1}{\delta}) & \text{for} \quad \frac{1}{\delta} < r < \infty. \end{cases}$$

Set $\mathcal{M}_\delta := \tau(\tilde{\mathcal{M}})$. Then $\mathcal{M}_\delta$ is relative compact, and for $f \in \mathcal{M}$ there holds

$$\begin{aligned}
&\|f - \tau(f|_K)\|_{-1+\varepsilon,-1-\varepsilon} \\
&= \max\left\{\sup_{0\le r\le\delta} r^{-1+\varepsilon}|f(r) - \frac{r}{\delta}f(\delta)|,\ \sup_{\frac{1}{\delta}\le r<\infty} r^{-1-\varepsilon'}|f(r) - r\delta f(\frac{1}{\delta})|\right\} \\
&\le \max\left\{\delta^\varepsilon \sup_{0\le r\le\delta}|\frac{f(r)}{r} - \frac{f(\delta)}{\delta}|,\ \delta^{\varepsilon'}\sup_{\frac{1}{\delta}\le r<\infty}|\frac{f(r)}{r} - \frac{f(\frac{1}{\delta})}{\frac{1}{\delta}}|\right\} \\
&\le 4\delta^{\min(\varepsilon,\varepsilon')}\|f\|_{-1,-1}.
\end{aligned}$$

This implies (5.3), as desired.

5.2 Uniqueness of weak radial solutions to a singular linear problem

The following Lemma provides uniqueness of weak solutions to a singular radially symmetric equation.

Lemma 5.2 *Consider $N \geq 3$, $\lambda \in \mathbb{R}$ and a locally bounded measurable function $g : (0,\infty) \to \mathbb{R}$ such that $\limsup\limits_{r\to 0} \frac{|g(r)|}{r^\alpha} < \infty$ for some $\alpha > -2$. Then the equation*

$$-\Delta u = g(|x|)u \qquad u(0) = \lambda$$

has at most one weak radial solution $u = u(r)$ in $W^{1,2}_{loc}(\mathbb{R}^N)$ such that $u \in C[0,\infty) \cap C^1(0,\infty)$ and u' is absolutely continuous on $(0,\infty)$.

Proof. If u is a solution in the above sense, then $r \mapsto r^{N-1}u'(r)$ is absolutely continuous on $(0,\infty)$ and there holds

$$(r^{N-1}u')' = -r^{N-1}g(r)u \tag{5.4}$$

in $L^1_{loc}(0,\infty)$. We first show

$$q := \lim_{r\to 0} r^{N-1}u'(r) = 0. \tag{5.5}$$

For this note that since $\limsup\limits_{r\to 0} |r^{N-1}g(r)u(r)| < \infty$, (5.4) implies that $r \mapsto r^{N-1}u'(r)$ is uniformly continuous near $r = 0$, hence the limit in (5.5) exists. Moreover

$$\begin{aligned}
q^2 &= \lim_{r\to 0} \frac{1}{r} \int_0^r (s^{N-1}u'(s))^2\, ds \\
&\leq \limsup_{r\to 0} r^{N-2} \int_0^r s^{N-1}(u'(s))^2\, ds \\
&= \limsup_{r\to 0} r^{N-2} \int_{B_r(0)} |\nabla u|^2 \\
&= \limsup_{r\to 0} r^{N-2} \|u\|^2 \\
&= 0,
\end{aligned}$$

hence $q = 0$. As a consequence, we can write u in the form

$$\begin{aligned}
u(r) &= \lambda - \int_0^r s^{-(N-1)} \int_0^s t^{N-1} g(t)u(t)\, dt\, ds \\
&= \lambda - \int_0^r u(s)g(s)h(s)\, ds \qquad (r > 0)
\end{aligned}$$

with $h(s) := s^{N-1} \int_s^r t^{-(N-1)}\, dt$. Now pick another solution v, and consider arbitrary $K > 0$. By assumption there exists $c > 0$ such that $|g(r)| \leq cr^{-\alpha}$ for $r \in [0, K]$. There holds

$$\begin{aligned}
|u(r) - v(r)| &\leq \int_0^r |u(s) - v(s)||g(s)||h(s)|\, ds \\
&\leq r \int_0^r |u(s) - v(s)||g(s)|\, ds,
\end{aligned}$$

since $|h(s)| \le r$ for $0 \le s \le r$. Now let

$$Y(r) := \int_0^r |u(s) - v(s)||g(s)|\, ds$$

and observe that $Y'(r) = |u(r) - v(r)||g(r)| \le cr^{1-\alpha}Y(r)$ for $0 < r \le K$. Therefore

$$\frac{d}{dr}\Big(Y(r)\exp(-\frac{c}{2-\alpha}r^{2-\alpha})\Big) \le 0$$

for $0 < r \le K$, which implies

$$Y(r)\exp\left(-\frac{c}{2-\alpha}r^{2-\alpha}\right) \le Y(s)\exp\left(-\frac{c}{2-\alpha}s^{2-\alpha}\right)$$

for $0 < s \le r \le K$. Passing to the limit $s \to 0$, we observe $Y(r) = 0$ for $0 < r \le K$. Since $K > 0$ was arbitrary, we conclude $u(r) = v(r)$ for all $r > 0$.

5.3 Courant's nodal domain theorem for a weighted eigenvalue problem

The purpose of this section is to prove a nodal property of eigenfunctions for a weighted linear eigenvalue problem of the form

$$-\Delta u = \lambda g(x)u \qquad u \in D^{1,2} := D^{1,2}(\mathbb{R}^N). \tag{5.6}$$

Again we denote by $J : D^{1,2} \to (D^{1,2})^*$ the canonical isometric isomorphism and $(\cdot|\cdot)$ the scalar product of $D^{1,2}$.

Theorem 5.3 *Let $g : \mathbb{R}^N \to \mathbb{R}$ denote a measurable a.e. nonnegative function which is bounded on compact subsets of $\mathbb{R}^N \setminus \Gamma$, where Γ is a closed subset of measure zero. Moreover, assume that for all $v, w \in D^{1,2}$ there exists*

$$\langle \tilde{g}v, w\rangle := \int_{\mathbb{R}^N} g(x)v(x)w(x) \tag{5.7}$$

and that the operator $\tilde{g} : D^{1,2} \to (D^{1,2})^*$ defined by (5.7) is compact.
Now if $u \in D^{1,2}$ is a continuous eigenfunction of the compact, nonnegative selfadjoint operator $G := J^{-1}\tilde{g} \in \mathcal{L}(D^{1,2})$ corresponding to the eigenvalue

$$\sigma_n := \sup_{\substack{V < X \\ \dim V = n}} \inf_{v \in V} \frac{(Gv|v)}{(v|v)} \tag{5.8}$$

such that $\sigma_n > 0$, then u has at most n nodal domains.

Remarks: (a) Note that the continuity of u implies that its nodal domains are well defined *open* subsets of $\mathbb{R}^N$.
(b) There is also a radial version of this Theorem. More precisely, assume that $g \in D_r^{1,2} := \{u \in D^{1,2} \mid u \text{ radially symmetric}\}$ and that (5.7) holds for $v, w \in D_r^{1,2}$. Then, if g is nonnegative and locally bounded on $\mathbb{R}^N \setminus \Gamma$ (Γ a closed subset of measure zero), and if the

corresponding operator $\tilde{g} : D_r^{1,2} \to (D_r^{1,2})^*$ is compact, then the assertion holds true for eigenfunctions of the operator $G := J^{-1}\tilde{g} \in \mathcal{L}(D_r^{1,2})$ as well. This follows since, as we shall see, all considerations below can be replaced by arguments respecting the rotational invariance.
(c) The *classical* nodal domain theorem of Courant asserts that the eigenfunction corresponding to the n-th eigenvalue of $-\Delta$ on a bounded domain $\Omega \subset \mathbb{R}^N$ has at most n nodal domains (cf. [8]). This result has been generalized to a wide class of semi-bounded differential operators (see e.g. [18],[2] and the references therein). However, the author is not aware of any work concerning weighted problems on $D^{1,2}$.

Proof of Theorem 5.3. Assume in contradiction that a continuous eigenfunction u of G corresponding to $\sigma_n > 0$ has at least $n+1$ nodal domains $\Omega_1, ..., \Omega_{n+1}$. As remarked already, all the Ω_i are *open* by the continuity of u. Define functions $v_i \in D^{1,2}$, $i = 1, ..., n$, $v_i \not\equiv 0$ by

$$v_i(x) := \begin{cases} u(x) & \text{for} \quad x \in \Omega_i \\ 0 & \text{for} \quad x \in \mathbb{R}^N \setminus \Omega_i \end{cases}$$

It is well known that the v_i are well defined elements of $D^{1,2}$ (cf. e.g. [13, p. 50]). Let V denote the span of $v_1, ..., v_n$. A direct calculation shows that for each $v \in V$ there holds

$$(Gv|v) = \sigma_n \|v\|^2. \tag{5.9}$$

Now choose orthonormalized eigenfunctions $u_1, ..., u_{n-1}$ of G corresponding to the eigenvalues $\sigma_1, ..., \sigma_{n-1}$, and let W denote the span of $u_1, ..., u_{n-1}$. Since $\dim W = n-1$, there exists $v \in V \cap W^\perp, v \neq 0$, which by (5.9) has to be an eigenfunction of G corresponding to σ_n. Moreover $\sigma_n > 0$ by assumption, hence v is a weak solution of

$$-\Delta v = \frac{1}{\sigma_n} gu. \tag{5.10}$$

However there holds $v(x) = 0$ for $x \in \Omega_{n+1}$, and this contradicts the unique continuation property of solutions to (5.10) as stated in [21, p. 519]. Thus the Theorem is proved.

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