Nonlinear Eigenvalue Problems Of Schrödinger Type
Admitting Eigenfunctions With Given Spectral Characteristics

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Abstract

The following work is an extension of our recent paper [10]. We still deal with nonlinear eigenvalue problems of the form

$$A_0 y + B(y)y = \lambda y \qquad (*)$$

in a real Hilbert space $\mathcal{H}$ with a semi-bounded self-adjoint operator A_0, while for every y from a dense subspace X of $\mathcal{H}$, $B(y)$ is a symmetric operator. The left–hand side is assumed to be related to a certain auxiliary functional ψ, and the associated linear problems

$$A_0 v + B(y)v = \mu v \qquad (**)$$

are supposed to have non-empty discrete spectrum $(y \in X)$.We reformulate and generalize the topological method presented by the authors in [10] to construct solutions of (*) on a sphere $S_R := \{y \in X | \, \|y\|_{\mathcal{H}} = R\}$ whose ψ-value is the n-th Ljusternik-Schnirelman level of $\psi|_{S_R}$ and whose corresponding eigenvalue is the n-th eigenvalue of the associated linear problem (**), where $R > 0$ and $n \in \mathbb{N}$ are given. In applications, the eigenfunctions thus found share any geometric property enjoyed by an n-th eigenfunction of a linear problem of the form (**). We discuss applications to elliptic partial differential equations with radial symmetry.

1 Introduction

This paper is a sequel to our recent paper [10] (cf. also [9] and [8]), in which we proposed a new topological method by which solutions to certain nonlinear eigenvalue problems can be constructed as fixed points of the solution operator of a family of associated *linear* eigenvalue problems. The advantage is that, for a given natural integer n, the eigenvalue corresponding to a solution thus constructed is known to be the n–th eigenvalue of an associated linear problem, so that our solution must share any geometric property (e. g. nodal structure) enjoyed by an n–th eigenfunction of the associated linear problem. At the same time, our solutions enjoy a variational characterization of Ljusternik–Schnirelman type, thus establishing an intimate relationship between a minimax characterization of solutions to nonlinear eigenvalue problems and the classical minimax characterizations of eigenvalues.

In the present paper, this method is considerably generalized, simplified, and improved, and some new applications are given. More specifically, we consider nonlinear eigenvalue problems which can be stated as operator equations of the form

$$A(y)y = \lambda y \tag{1.1}$$

in a Gelfand triplet $X \hookrightarrow H \hookrightarrow X^*$, together with the constraint

$$\|y\| = R. \tag{1.2}$$

Here H is a (real) Hilbert space, $\|\cdot\|$ is its norm, X is a densely embedded Banach space, and $A : X \to \mathcal{L}(X, X^*)$ is a continuous map satisfying some additional requirements which will be specified in Section 2. As a typical example, one should think of a boundary value problem for a differential equation of the form

$$Ly + f(x, y^2)y = \lambda y, \tag{1.3}$$

where L is an elliptic linear differential operator, and the second term on the left–hand side is an odd higher–order nonlinearity which has been conveniently written in the form $f(x, y^2)y$. (However, applications to systems of differential equations, to functional differential equations, or to equations with a nonlinearity on the right–hand side are also possible, and some such applications have been presented in the papers cited above.) The associated family of linear eigenvalue problems is given by

$$A(y)u = \mu(y)u, \tag{1.4}$$

where y ranges through

$$S_R := \{y \in X |\; \|y\| = R\},$$

and we assume that the operators $A(y) : X \to X^*$ are induced by *self–adjoint* operators

$$\tilde{A}(y) = \tilde{A}_0 + \tilde{B}(y),$$

where $\tilde{A}_0$ is a fixed self–adjoint operator of Schrödinger type in H, and where the $\tilde{B}(y)$ are symmetric perturbations. The term 'Schrödinger type' refers to the assumption that the spectrum of $\tilde{A}_0$ is of the form

$$\sigma(\tilde{A}_0) = \{\mu_1, \mu_2, \dots\} \cup \sigma_{\mathrm{ess}}(\tilde{A}_0),$$

where the μ_n are (finitely or infinitely many) isolated eigenvalues of finite multiplicity such that $\mu_n < \mu_\infty$, where μ_∞ is the infimum of the essential spectrum (which may be empty, in which case we put $\mu_\infty := \infty$). The perturbations under consideration are harmless enough not to alter this picture. Thus, for each $y \in X$ we have the increasing (finite or infinite) sequence of eigenvalues $\mu_n(y)$ of $\tilde{A}(y)$, counted with multiplicity, while the essential spectrum does not depend on y. For fixed $n \in \mathbb{N}$ we put

$$K = K_{n,R} := \{y \in S_R | y \in V_n(y)\},$$

where $V_n(y)$ is the span of all the eigenvectors of $\tilde{A}(y)$ corresponding to eigenvalues $\mu_k(y)$ with $k \leq n$.

To find solutions of the original problem in K we need an auxiliary functional $\varphi : X \to \mathbb{R}$ satisfying a certain 'comparison condition' (cf. condition (CC) below). In most applications the problem has variational structure, and φ is a C^1–functional whose derivative is the nonlinearity. For instance, for eq. (1.3) one can take

$$\varphi(y) := \frac{1}{2} \int F(x, y(x)^2)\, dx \qquad \text{with} \qquad F(x,s) := \int_0^s f(x,t)\, dt,$$

and the comparison condition then amounts to the requirement that $f(x, \cdot)$ should be nondecreasing on $[0, \infty[$. We then put

$$\psi(y) := \frac{1}{2}\langle A_0 y, y\rangle + \varphi(y),$$

and we consider the Ljusternik–Schnirelman levels

$$c_k = c_k(R) := \inf_{C \in \Sigma_k(R)} \sup_{y \in C} \psi(y) \qquad (k \in \mathbb{N}),$$

where $\Sigma_k(R)$ denotes the system of all closed symmetric subsets C of S_R whose Krasnosel'skij genus $\gamma(C)$ (taken with respect to the topology of X) is at least k. The main result now is that under reasonable assumptions one can establish the following crucial property:

(CP) K is compact and nonempty, $\gamma(K) = n$, $c_n = \max_{u \in K} \psi(u)$, and every $y \in K \cap \psi^{-1}(c_n)$ is a solution of equation (1.1) with eigenvalue $\lambda = \mu_n(y)$.

In Section 2 of the present paper we establish property (CP) for a much wider class of problems than before. The main improvements are:

(i) The presence of essential spectrum (i. e. $\mu_\infty < \infty$) is now admitted.

(ii) In its present form, the general theory can be applied to differential equations directly, without the cumbersome cut–off procedures which had to be developped for each application separately in [10] and [8].

(iii) The functional φ is only required to be continuous, and its only relationship to the given problem is condition (CC). Therefore the treatment of non–variational problems by the present method is now within reach.

(iv) The perturbation theory is essentially carried out in $\mathcal{L}(X, X^*)$. This enables us to admit more general perturbations than relatively $\tilde{A}_0$–compact ones.

As a corollary, we derive a very general multiplicity result for Ljusternik–Schnirelman levels at the end of Section 2.

The power of the abstract theory is illustrated in Section 3 by applying it to radially symmetric (stationary) nonlinear Schrödinger equations on $\mathbb{R}^N$ $(N \geq 2)$. In other words, we consider eq. (1.3) with $L := -\Delta + q$, where q is a radially symmetric attractive potential vanishing at infinity, and with a nonlinearity of the form $f(|x|, y^2)y$, where $f(r, 0) \equiv 0$ and $f(r, \cdot)$ is nondecreasing for every $r > 0$. Both q and f may be singular at the origin, but they have to satisfy appropriate growth restrictions at zero and infinity. For every $n \in \mathbb{N}$ and $R > 0$, we then establish property (CP) and hence, in particular, the existence of a radially symmetric solution $u_{n,R}$ having precisely $n - 1$ nodal surfaces and satisfying $\int u_{n,R}^2 \, dx = R^2$. Under somewhat weaker hypotheses our result still holds for R sufficiently small. Note that the equations treated here are in the *sublinear* case because our assumptions on f imply $f(|x|, s) \geq 0$, and it seems that for this case very little is known about nodal solutions, while the superlinear case has been studied extensively by various authors (see e. g. Bartsch and Willem [3] and the references there). A related sublinear equation involving a convolution term has been treated by Stuart [20, 21, 22], who used a generalized mapping degree method, and by Lions [15], who used variational methods, and who also comments on the difficulties encountered with this approach (cf. [15], p. 37f.). Finally, Bongers, Heinz and Kpper [4, 11, 12, 13] have established the existence of multiple and nodal solutions for a class of sublinear equations of type (1.3) on unbounded domains, but the nonlinearities in their equations are required to grow strongly at infinity, whereas in the equations treated in the present paper, the data functions decay at infinity.

The material of Section 3 is supplemented in Section 4 by considering the same type of equations on other radially symmetric domains, together with Dirichlet boundary conditions. In all cases (i. e. for balls, complements of balls, and annuli) it turns out that property (CP) and its consequences hold true under appropriate weaker assumptions on q and f. One can simply remove the growth restrictions which were imposed to take care of the singularity at zero (resp. at infinity) when the respective singular point does not appear in the domain under consideration. The presentation is brief and sketchy here, because the proofs are straightforward simplifications of the proofs given in Section 3. Nevertheless, these results generalize those of Heid [8], removing the undesirable technical assumptions for low dimensions appearing there (cf. Remark a) in Section 6 of [10]).

2 Abstract Nonlinear Eigenvalue Problems

In this section we develop the abstract framework for the treatment of problem (1.1)-(1.2) and we present a set of conditions which are sufficient for (CP) to hold.
Let $\mathcal{H}$ denote a real Hilbert space with scalar product $(\cdot|\cdot)$ and norm $\|\cdot\|$, and let $A_0 : \mathcal{D}(A_0) \to \mathcal{H}$ be an (unbounded) linear operator in $\mathcal{H}$ which is self–adjoint and bounded from below. For convenience we assume

$$A_0 \geq I \tag{2.1}$$

throughout this section. At the end of this section we show how our results can be generalized to other lower bounds.
Let X be the form domain of A_0, and let X^* be its topological dual. Since the range of the inclusion $i : X \to \mathcal{H}$ is dense in $\mathcal{H}$, the canonical identification of $\mathcal{H}$ with its dual leads to the following embeddings:

$$X \overset{i}{\hookrightarrow} \mathcal{H} \overset{i^*}{\hookrightarrow} X^*$$

We therefore regard all the vector spaces defined above as subspaces of X^*. In particular that means that if $v \in \mathcal{H}$, we refer to v also as an element of X^* instead of writing i^*v.
As a consequence of (2.1), the operator $W := \sqrt{(}A_0)$ is well defined and selfadjoint with domain $\mathcal{D}(W) = X$, and X becomes a Hilbert space with the scalar product $(u, v)_X := (Wu, Wv)$ for all $u, v \in X$. When we speak of the Hilbert space X, we always refer to this inner product. Denoting by $W^* : H \to X^*$ the dual map of W, the canonical isometric isomorphism $J : X \to X^*$ is given by $J = W^*W$, since for $u, v \in X$ we have

$$\langle W^*Wu, v\rangle = (Wu, Wv) = (u, v)_X,$$

where $\langle\cdot,\cdot\rangle : X^* \times X \to \mathbb{R}$ denotes the dual pairing. Moreover, the natural Hilbert space structure of X^* comes from the inner product

$$(u, v)_{X^*} := (W^{*-1}u, W^{*-1}v).$$

Passing to the induced norms, there holds

$$\|u\|_X \geq \|u\| \qquad \forall\, u \in X$$

and

$$\|u\| \geq \|u\|_{X^*} \qquad \forall\, u \in H.$$

Since some of our arguments refer to the Hilbert spaces X or X^* rather then H, it will be useful to distinguish between corresponding notions. ¿From now on we shall therefore talk about H-selfadjoint operators versus X^*-selfadjoint operators or, e.g., X-orthogonal vectors versus H-orthogonal vectors. Speaking in those terms, one easily observes the following:

Lemma 2.1 *The operator J defined above with domain $\mathcal{D}(J) = X \subset X^*$ is X^*-selfadjoint. Moreover, $\sigma(J) = \sigma(A_0)$.*

Proof : We recall that $W^* : H \to X^*$ is an isometric isomorphism between Hilbert spaces. Therefore we just need to see that it maps $\mathcal{D}(A_0)$ onto $\mathcal{D}(J)$ and that the relation

$$J = W^* A_0 W^{*^{-1}} : X \to X^*$$

holds. But this follows from the observation that $W^*v = Wv$ for all $v \in \mathcal{D}(A_0)$ and that W maps $\mathcal{D}(A_0)$ onto $\mathcal{D}(J)$.

Now we shall consider equations of the form

$$A(y)y = \lambda y$$

together with the constraint

$$\|y\| = R,$$

with $R > 0$ given. Here the family of operators $(A(y))_{y\in X}$ is built up as follows: For each $y \in X$ we define $A(y)$ as the form sum $A_0 + \tilde{B}(y)$, where $\tilde{B}(y)$ is a symmetric operator in $\mathcal{H}$ with the following properties:

(H0) For every $y \in X$ the domain $\mathcal{D}(\tilde{B}(y)) \subset X$ is dense, and the map $i^*\tilde{B}(y) : \mathcal{D}(\tilde{B}(y)) \to X^*$ extends to a continuous linear operator $B(y) : X \to X^*$.

This gives rise to a (nonlinear) map $B : X \to \mathcal{L}(X, X^*)$, which is supposed to satisfy the fundamental hypotheses:

(H1) B is completely continuous, i.e. compact and continuous.

(H2) $B(y) \in \mathcal{L}(X, X^*)$ is a compact operator for every $y \in X$.

(H3) $B(0) = 0$, and moreover $B(-y) = B(y)$ for all $y \in X$.

(CC) ('Comparison Condition') There is a continuous map $\varphi : X \to \mathbb{R}$ such that for arbitrary vectors $y, v \in X$ there holds

$$2(\varphi(v) - \varphi(y)) \geq \langle B(y)v, v\rangle - \langle B(y)y, y\rangle.$$

Note that these assumptions are weaker then the ones of [10]and [8]. In particular, the $B(y)$ do not have to be bounded linear operators in H (but may be of course). Moreover, we do *not* need a factorisation for B of the form $X \hookrightarrow C \to \mathcal{L}(X, X^*)$ with a compact embedding of X in some Banach space C. Furthermore we do not require that the functional φ in condition (CC) is a primitive of the nonlinear operator $y \mapsto B(y)y$, but in the applications we have in mind, this actually is true. Finally and most important, we drop the condition (GB) from [10], so now the whole theory is much easier to handle in applications to differential equations.

In view of (CC) we can normalize φ so as to have

$$\varphi(0) = 0. \tag{2.2}$$

As in [10] we define $\psi : X \to \mathbb{R}$ by

$$\psi(y) := \frac{1}{2}\|y\|^2 + \varphi(y) \qquad\qquad (y \in X).$$

For arbitrary $R, \tilde{R} > 0$ we use the notations

$$\begin{aligned} S_R &:= \{y \in X |\, \|y\| = R\}, \\ T_R &:= \{y \in X |\, \|y\| \leq R\}, \\ D_{\tilde{R}} &:= \{y \in X |\, \|y\|_X \leq \tilde{R}\}, \\ T(R, \tilde{R}) &:= T_R \cap D_{\tilde{R}}. \end{aligned}$$

Note that, in general, S_R and T_R are unbounded in X. As usual, we set

$$c_n(R) := \inf_{A \in \Sigma_n(R)} \sup_{y \in A} \psi(y).$$

We have not yet shown that, by virtue of our assumptions, $A(y)$ is well defined as a form sum for every $y \in X$. Not only for this purpose the following *uniform* estimates for the corresponding quadratic form defined by

$$[v, w]_y := (v, w)_X + \langle B(y)v, w\rangle \qquad \forall\, y, v, w \in X$$

turn out to be useful:

Lemma 2.2 *Consider a bounded subset $D \subset X$. Then there are positive constants $a, b, c \in \mathbb{R}$ such that*

$$a\|v\|_X^2 \leq [v, v]_y + c\|v\|^2 \leq b\|v\|_X^2 \tag{2.3}$$

for all $y \in D$, $v \in X$.

Proof: First we claim that for each $\varepsilon > 0$ there exists a $K := K(D)$ such that

$$\|B(y)v\|_{X^*} \leq \varepsilon\|v\|_X + K\|v\|_{X^*} \qquad \forall y \in D,\ v \in X. \tag{2.4}$$

Assuming to the contrary that this does not hold for some $\varepsilon > 0$, we would find sequences $(y_n)_n \subset D$ and $(v_n)_n \subset X$ with

$$\|B(y_n)v_n\|_{X^*} > \varepsilon\|v_n\|_X + n\|v_n\|_{X^*} \qquad \forall n \tag{2.5}$$

Because of conditions (H1) and (H2) we may assume that, passing to a subsequence, $(B(y_n))_n$ converges to a *compact* operator $T \in \mathcal{L}(X, X^*)$. Thus T, viewed as a closed operator in X^*, is relatively $J - compact$, since $\|Jv\|_{X^*} = \|v\|_X$ for $v \in X$. The *Theorem of Ehrling* now provides a constant K such that

$$\|Tv\|_{X^*} \leq \frac{\varepsilon}{2}\|v\|_X + K\|v\|_{X^*}$$

for all $v \in X$. Therefore we get

$$\begin{aligned} \|B(y_n)v_n\|_{X^*} &\leq \|(B(y_n) - T)v_n\|_{X^*} + \|Tv_n\|_{X^*} && (2.6)\\ &\leq (\|B(y_n) - T\|_{\mathcal{L}(X,X^*)} + \frac{\varepsilon}{2})\|v_n\|_X + K\|v_n\|_{X^*}. && (2.7) \end{aligned}$$

This contradicts (2.5) for sufficiently large n, and (2.4) is proven. Next we prove that there are $a, c > 0$ such that the left hand inequality of (2.3) holds. Taking $0 < \varepsilon < \frac{1}{2}$ in (2.4), we have

$$\begin{aligned} \|v\|_X^2 &= [v, v]_y - \langle B(y)v, v\rangle && (2.8)\\ &\leq [v, v]_y + \|B(y)v\|_{X^*}\|v\|_X && (2.9)\\ &\leq [v, v]_y + \varepsilon\|v\|_X^2 + K\|v\|_{X^*}\|v\|_X. && (2.10) \end{aligned}$$

Since

$$\|v\|_{X^*}\|v\|_X \leq \frac{\varepsilon}{K}\|v\|_X^2 + \frac{K}{\varepsilon}\|v\|_{X^*}^2 \leq \frac{\varepsilon}{K}\|v\|_X^2 + \frac{K}{\varepsilon}\|v\|^2,$$

we get

$$a\|v\|_X^2 \leq [v, v]_y + c\|v\|^2$$

with $a := 1 - 2\varepsilon$ and $c := \frac{K^2}{\varepsilon}$. Finally, using again (2.4), we obtain

$$\begin{aligned} [v, v]_y &\leq \|v\|_X^2 + \|B(y)v\|_{X^*}\|v\|_X && (2.11)\\ &\leq \|v\|_X^2 + (\varepsilon\|v\|_X + K\|v\|_{X^*})\|v\|_X && (2.12)\\ &\leq (1 + \varepsilon + K)\|v\|_X^2, && (2.13) \end{aligned}$$

since $\|v\|_{X^*} \leq \|v\| \leq \|v\|_X$ for $v \in X$. Therefore

$$[v, v]_y + c\|v\|^2 \leq b\|v\|_X,$$

setting $b := 1 + \varepsilon + K + c$, and this completes the proof.

Corollary 2.3 *For each $y \in X$, the quadratic form $[\cdot,\cdot]_y$ is closed and halfbounded in H with domain X and continuous from $X \times X$ to $\mathbb{R}$. In particular, there is a unique H–selfadjoint semi–bounded operator $A(y)$ such that $\mathcal{D}(A(y)) \subset X$ and $(A(y)v, v) = [v, v]_y$ for all $v \in \mathcal{D}(A(y))$.*

Note furthermore that, by Lemma 2.2, the operator $J(y) := J + B(y)$ is closed in X^* with domain $\mathcal{D}(J(y)) = X \subset X^*$ for every $y \in X$. As one might expect, there are relationships between $A(y)$ and $J(y)$. The following lemma ensures that $A(y)$ and $J(y)$ are quite similar from the 'spectral theoretic' point of view.

Lemma 2.4 *Consider an arbitrary $y \in X$. Then there holds*

(a) $\sigma(A(y)) = \sigma(J(y))$, $\sigma_p(A(y)) = \sigma_p(J(y))$ and $\sigma_c(A(y)) = \sigma_c(J(y))$. Furthermore $\sigma_r(J(y)) = \emptyset$.

(b) Pick $\lambda \in \sigma_p(A(y))$. Then $P'(X^*) = P(H) \subset \mathcal{D}(A(y))$, where P resp. P' denote the eigenprojection associated to λ and the operator $A(y)$ resp. $J(y)$.

Proof: Fix $y \in X$ and chose a number $c > 0$ such that $A(y) + cI_H \geq I_H$. Writing $\tilde{A} := A(y) + cI_H$ as well as $\tilde{J} := J(y) + cI_{X^*}$, it obviously suffices to prove the statements for $\tilde{A}$ and $\tilde{J}$ in place of $A(y)$ and $J(y)$. To this end, we first recall that the domain of the positive operator $V := \sqrt{\tilde{A}}$ coincides with the form domain of $\tilde{A}$, i.e. with X, and that

$$(\tilde{A}v, w) = (Vv, Vw) \qquad \forall v, w \in \mathcal{D}(\tilde{A}) = \mathcal{D}(A(y)).$$

Thus, for all $v, w \in \mathcal{D}(A(y))$, we get

$$\begin{aligned} (Vv, Vw) &= (A(y)v|w) + c(v|w) \\ &= [v, w]_y + c(v|w) \\ &= \langle (J(y) + cI_{X^*})v, w\rangle \\ &= \langle \tilde{J}v, w\rangle. \end{aligned}$$

Actually this is even true for $v, w \in X$, since $\mathcal{D}(A(y)) \subset X$ is dense and both sides of the equality are continous in the norm of X. Introducing once again the dual operator $V^* : H \to X^*$, this just means $\tilde{J} = V^*V$. But $V : X \to H$ is a topological isomorphism between Hilbert spaces, so that $V^* : H \to X^*$ is one as well. Moreover we observe that

$$\tilde{J} = V^*\tilde{A}V^{*-1}, \tag{2.14}$$

because $V(\mathcal{D}(\tilde{A})) = X$ and $V^*|_{\mathcal{D}(\tilde{A})} = V$ (We are still considering H as a subspace of X^*). The relation (2.14) directly implies $\sigma(\tilde{A}) = \sigma(\tilde{J})$ and $\sigma_r(\tilde{J}) = \sigma_r(\tilde{A}) = \emptyset$, since

$\tilde{A}$ is selfadjoint.
Now consider an arbitrary $\lambda \in \mathbb{R}$ and the spectral projections P resp. P' associated to the set $\{\lambda\}$ and the operators $\tilde{A}$ resp. $\tilde{J}$. By (2.14) we observe that, if λ is an eigenvalue of $\tilde{A}$, then V maps $P(H)$ one-to-one into $P'(X^*)$, and in particular λ is an eigenvalue of $\tilde{J}$. On the other hand, if λ is an arbitrary eigenvalue of $\tilde{J}$, then (2.14) also implies that V^{-1} maps $P'(X^*)$ one-to-one into $P'(H)$, therefore λ must be an eigenvalue of $\tilde{A}$. We conclude that $\sigma_p(\tilde{A}) = \sigma_p(\tilde{J})$, $\sigma_c(\tilde{A}) = \sigma_c(\tilde{J})$ and that for $\lambda \in \sigma_p(\tilde{A})$ we have

$$V : P(H) \xrightarrow{\cong} P'(X^*)$$

(P, P' defined as above). However, the restriction of V to $P(H)$ is just the multiplication with $\lambda^{\frac{1}{2}} > 0$, hence these spaces are identical. This completes the proof.

Under the assumptions we have introduced so far, the operators $A(y)$, $y \in X$ do not necessarily have any discrete spectrum at all. To describe the additional assumptions needed to circumvent this problem, we define the nondecreasing sequence of values

$$\mu_k(y) := \inf_{\substack{V \leq X \\ \dim V = k}} \sup_{v \in V} \frac{[v,v]_y}{(v,v)},$$

for each $y \in X$ and set

$$\mu_\infty := \inf \sigma_{ess}(A(y))$$

with the additional convention $\mu_\infty = \infty$ if $\sigma_{ess}(A(y))$ is void. (Note that $\sigma_{ess}(A(y))$ does not depend on y, because for all $y \in X$ we have $\sigma_{ess}(A(y)) = \sigma_{ess}(J(y))$ by the preceeding lemma, and $J(y)$ arises from J by a compact perturbation so that the essential spectrum is not changed.)
Fixing now $y \in X$, only two different cases occur (see e.g. [6]):

(a) $\mu_k(y) < \mu_\infty$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \mu_k(y) = \mu_\infty$. Moreover, all the $\mu_k(y)$ are eigenvalues of $A(y)$ each repeated a number of times equal to its multiplicity.

(b) There is a number $k_0 \in \mathbb{N}$ such that $\mu_{k_0}(y) < \mu_\infty$ and $\mu_k(y) = \mu_\infty$ for $k > k_0$. Then $\mu_1(y), ..., \mu_{k_0}(y)$ are eigenvalues of $A(y)$ each repeated a number of times equal to its multiplicity.

To deal with this situation, we now introduce our final abstract condition which depends on $n \in \mathbb{N}$ as well as a subset $D \subset X$:

$(EC)_{n,D}$ There holds $\mu_n(y) < \mu_{n+1}(y)$ for every $y \in D$ and moreover

$$\sup_{y \in D} \mu_n(y) < \mu_\infty \tag{2.15}$$

If $(EC)_{n,D}$ holds true, then for $y \in D$ the numbers $\mu_1(y), ..., \mu_n(y)$ are eigenvalues of $A(y)$, and we can define the subspace $V_n(y)$ as the span of the corresponding eigenvectors. But even if this is not the case, we still define the space $V_n(y)$ for *every* $y \in X$ as

$$V_n(y) := \left\{ \sum_{k=1}^{n} \alpha_k u_k \,|\alpha_k \in \mathbb{R}, u_k \in \mathcal{D}(A(y)), A(y)u_k = \mu_k(y)u_k \quad (k = 1, \ldots, n) \right\}.$$

On the other hand, in case $(EC)_{n,D}$ is satisfied, Lemma 2.4 provides the identity

$$V_n(y) = \left\{ \sum_{k=1}^{n} \alpha_k u_k \,|\alpha_k \in \mathbb{R}, u_k \in X, J(y)u_k = \mu_k(y)u_k \quad (k = 1, \ldots, n) \right\}$$

for $y \in D$.

We now have the tools to prove a generalization of proposition 3.3 of [10], which will pave the way for the application of the abstract projection method (cf. [10]).

Proposition 2.5 *Suppose (H1), (H2) are satisfied, and consider $n \in \mathbb{N}, D \subset X$ such that $(EC)_{n,D}$ holds. For each $y \in D$, let $P_n(y)$ be the X-orthogonal projector of X onto $V_n(y)$. Then we have:*

(a) The map $P_n : D \to \mathcal{L}(X)$ is continuous.

(b) If D is bounded in X, then the range $P_n(D)$ is relatively compact in $\mathcal{L}(X)$.

Proof: We can use some of the arguments from [10], with more or less slight changes:
(a) Let us first consider the X^*-orthogonal projector $Q_n(y)$ of X^* onto $V_n(y)$ for $y \in D$, and let us prove that the map

$$Q_n : D \to \mathcal{L}(X^*, X)$$

is continuous. This can be shown in a very similar way as in [10]. More precisely, let $D_0 \subseteq D$ be a bounded neighbourhood of an arbitrary point $y_0 \in D$ and choose positive constants a, b, c such that (2.3) holds for all $y \in D_0$. Then, if $y_1, y_2 \in D_0$, $0 < \varepsilon < a$ are such that

$$\|B(y_1) - B(y_2)\| < \varepsilon, \tag{2.16}$$

we have

$$\begin{aligned} [v,v]_{y_2} &= [v,v]_{y_1} + \langle (B(y_2) - B(y_1))v, v\rangle \\ &\leq [v,v]_{y_1} + \varepsilon \|v\|_X^2 \\ &\leq [v,v]_{y_1} + \frac{\varepsilon}{a}\left([v,v]_{y_2} + c\|v\|^2\right) \end{aligned}$$

and hence

$$\frac{a-\varepsilon}{a}[v,v]_{y_2} \le [v,v]_{y_1} + \frac{c\varepsilon}{a}\|v\|^2$$

for all $v \in X$. It now follows from the definition of the μ_k that

$$\mu_k(y_2) \le \frac{a}{a-\varepsilon}\mu_k(y_1) + \frac{c\varepsilon}{a-\varepsilon} \le \mu_k(y_1) + \frac{\varepsilon}{a-\varepsilon}(M+c)$$

for $1 \le k \le n$, where M is an upper bound for the $\mu_k(y),\ k=1,\dots,n,\ y \in D$ (which exists because of assumption $(EC)_{n,D}$). Interchanging the roles of y_1 and y_2 we see that (2.16) implies

$$|\mu_k(y_1) - \mu_k(y_2)| \le \frac{\varepsilon}{a-\varepsilon}(M+c)$$

for $k=1,\dots,n$. Therefore (H1) entails continuous dependance of the $\mu_1(y),\dots,\mu_n(y)$ on $y \in D$ with respect to the topology of X.
Fixing $y_0 \in D$ again, we find a closed Jordan curve Γ in the complex plane and a $\delta > 0$ such that for y in $B_\delta(y_0)$ we have

$$Q_n(y_0) = \frac{1}{2\pi i}\int_\Gamma (\lambda I - J(y))^{-1}\, d\lambda. \tag{2.17}$$

(Actually this is true for the complexification of $Q_n(y_0)$, but for the sake of brevity we do not express this in the notation.) The continuity of Q_n now follows from the fact that the map

$$\Gamma \times B_\delta(y_0) \to \mathcal{L}(X^*, X) : (\lambda, y) \mapsto (\lambda I - J(y))^{-1}$$

is continuous. To establish this fact, note first that the norms $\|(\lambda I - J(y))^{-1}\|_{\mathcal{L}(X^*,X^*)}$ remain bounded on $\Gamma \times B_\delta(y_0)$ by spectral theory. Moreover, if we choose $a, b, c \in]0,\infty[$ such that (2.3) holds on D, the relation

$$[v,v]_y + c\|v\|^2 = \langle (J(y)+cI)v, v\rangle$$

together with (2.3) shows that we have

$$\|(J(y)+cI)v\|_{X^*} \ge a\|v\|_X$$

for all $v \in X,\ y \in D$. Since $\sigma_r(J(y)) = \emptyset$ by Lemma 2.4, this implies that the operators $J(y)+cI,\ y \in D$ form a family of isomorphisms $X \to X^*$ such that the norms

$$\|(J(y)+cI)^{-1}\|_{\mathcal{L}(X^*,X)}$$

remain bounded by a^{-1}. Since

$$(\lambda I - J(y))^{-1} = -(J(y)+cI)^{-1}(I - (c+\lambda)(\lambda I - J(y))^{-1}),$$

it follows that

$$\sup_{\lambda\in\Gamma,\, y\in B_\delta(y_0)} \|(\lambda I - J(y))^{-1}\|_{\mathcal{L}(X^*,X)} < \infty.$$

Hence the desired continuity result follows from (H1) and the relation

$$\begin{aligned} & (\lambda_1 I - J(y_1))^{-1} - (\lambda_2 I - J(y_2))^{-1} = \\ = \; & (\lambda_2 I - J(y_2))^{-1}((\lambda_2 - \lambda_1)I - (B(y_2) - B(y_1)))(\lambda_1 I - J(y_1))^{-1} \end{aligned}$$

for $y_1, y_2 \in B_\delta(y_0),\ \lambda_1, \lambda_2 \in \Gamma$.

Since $X \subset X^*$, the set

$$\{(y,v) \in D \times X | v \in \mathcal{R}(Q_n(y))\}$$

is a continuous sub-bundle of the trivial bundle $D \times X$. However, $\mathcal{R}(Q_n(y)) = V_n(y)$, so that $P_n : D \to \mathcal{L}(X)$ is given as the pointwise X-orthogonal projection on the fibres of this bundle which is a continuous map by the theory of vector bundles. Thus (a) is proved.

For (b) we only have to show that for $k = 1, ..., n$ the set

$$E_k := \{u |\ u \in \mathcal{D}(A_y),\ \|u\| = 1 \text{ and } u \in \mathcal{N}(\mu_k(y)I - A(y)) \text{ for some } y \in D\}$$

is relatively compact in X, since then the relative compactness of $P_n(D)$ can then be derived exactly as in [10].

So let us pick $k \in \{1, ..., n\}$ and consider $u \in E_k$. Then

$$[u,u]_y = (A_y u, u) = \mu_k(y) \le K < \infty, \tag{2.18}$$

with a constant $K > 0$, since $(EC)_{n,D}$ is satisfied. Therefore

$$\|u\|_X \le \frac{1}{a}(K + c)$$

with a and c chosen as in Lemma 2.2. Thus, E_k is bounded in X. ¿From Lemma 2.2 we also deduce that the operators $A(y)$ are bounded from below *uniformly* for $y \in D$, hence

$$c_- := \inf_{y\in D} \mu_k(y) > -\infty.$$

Moreover, as a consequence of $(EC)_{n,D}$, there is an $\varepsilon > 0$ such that

$$\mu_k(y) < \mu_\infty - \varepsilon \quad \text{for all } y \in D$$

(In case $\mu_\infty = \infty$ one should replace this by a number $K > 0$ instead of $\mu_\infty - \varepsilon$). We recall that J only has finitely many eigenvalues (counted with multiplicity) smaller or equal $\mu_\infty - \varepsilon$. Thus, there is a *compact* selfadjoint operator G in X^* (even one of finite rank) such that the spectrum $\sigma(J+G)$ of the X^*-selfadjoint operator $J+G$ lies completely above $\mu_\infty - \varepsilon$. Denoting $R(\mu) : X^* \to X$ the resolvent of $J+G$, for $u \in E_k$ there holds the relation

$$u = R(\mu_k(y))(B(y) - G)u \tag{2.19}$$

with an appropriate $y \in D$. However, we observe that, as a consequence of (H1), (H2), the nonlinear operator $B_G : X \times X \to X^*$ defined by

$$(z, v) \overset{B_G}{\longmapsto} (B(z) - G)v$$

is compact, hence $B_G(D \times E_k)$ is relatively compact in X^*. Moreover, the map

$$\tilde{R} : [c_-, \mu_\infty - \varepsilon] \times X^* \to X, \qquad (\mu, w) \mapsto R(\mu)w$$

is continuous. Thus, $\tilde{R}\Big([c_-, \mu_\infty - \varepsilon] \times B_G(D \times E_k)\Big)$ is relatively compact in X, and it contains E_k as a subset because of (2.19). Therefore E_k is relatively compact as well, and that completes the proof.

Next we recall some basic consequences of (CC) similar to those stated in [10]. First, the choice $v = 0$ in (CC) leads to

$$\varphi(y) \le \frac{1}{2}\langle B(y)y, y\rangle \qquad \forall\, y \in X. \tag{2.20}$$

Moreover, using (H3) and taking $y = 0$ in (CC), we obtain

$$\varphi(v) \ge 0 \qquad (v \in X). \tag{2.21}$$

Hence for the functional ψ there holds

$$\psi(v) \ge \frac{1}{2}\|v\|_X^2, \tag{2.22}$$

which in particular implies $c_n(R) > -\infty$ for all $R > 0$, $n \in \mathbb{N}$.

Lemma 2.6 *Suppose hypotheses (H1) – (H3) and (CC) are satisfied. Let $R > 0$ and consider $n \in \mathbb{N}$ and $y \in T_R$. If $y \in V_n(y)$, then:*

(a) $\psi(y) \le c_n(R)$

(b) If $\mu_n(y)$ is not an eigenvalue of $A(y)$, then $\psi(y) < c_n(R)$.

Proof: Fix $y \in X$ satisfying the assumptions of the proposition. We first consider the case that $\mu_n(y)$ is an eigenvalue of $A(y)$(which implies that $\mu_1(y), ..., \mu_{n-1}(y)$ are eigenvalues of $A(y)$, too). We define the Rayleigh quotient $\rho : X \setminus \{0\} \to \mathbb{R}$ by

$$\rho(u) := \frac{\langle A(y)u, u\rangle}{\|u\|^2}.$$

Now (CC) implies

$$\psi(v) - \psi(y) \geq \frac{R^2}{2}\rho(v) - \frac{\|y\|^2}{2}\rho(y)$$

for every $v \in S_R$. Choose pairwise orthogonal eigenvectors $u_1, \ldots, u_{n-1}$ corresponding to $\mu_1(y), \ldots, \mu_{n-1}(y)$, and let $W \subseteq V_n(y)$ be the span of $u_1, \ldots, u_{n-1}$. Then clearly

$$\mu_n(y) = \inf_{v \in S_R \cap W^\perp} \rho(v),$$

and, moreover, $y \in V_n(y)$ implies $\rho(y) \leq \mu_n(y)$. Thus, denoting $W^\perp := \{v \in X \mid (v|w) = 0 \, \forall w \in W\}$, we obtain

$$\begin{aligned} \inf_{v \in S_R \cap W^\perp} \psi(v) - \psi(y) &\geq \frac{R^2}{2}\mu_n(y) - \frac{\|y\|^2}{2}\rho(y) \\ &\geq (\frac{R^2}{2} - \frac{\|y\|^2}{2})\rho(y) \\ &\geq 0. \end{aligned} \tag{2.23}$$

In the last step we have combined (2.20), (2.21) and the fact that $\|y\| \leq R$ by assumption. But, as it has been stated in [10, proof of prop. 3.2], there holds

$$\inf_{v \in S_R \cap W^\perp} \psi(v) \leq c_n(R), \tag{2.24}$$

and consequently

$$c_n(R) - \psi(y) \geq 0.$$

Thus (a) is proved.

To prove (b), consider the case that $\mu_n(y)$ is not an eigenvalue of $A(y)$, hence $\mu_n(y) = \mu_\infty \in \sigma_c(A(y))$. We then nevertheless have $V_n(y) = V_m(y)$, where $\mu_m(y) < \mu_n(y)$ is the largest eigenvalue below $\sigma_c(A(y))$. (If A(y) has no eigenvalue below $\sigma_c(A(y))$, then $V_n(y) = \{0\}$, and the assertion is trivial). Clearly we now have

$$\mu_n(y) = \inf_{v \in S_R \cap V_n(y)^\perp} \rho(v)$$

and $y \in V_n(y)$ implies $\rho(y) \leq \mu_m(y) < \mu_n(y)$. In the same way as in (a) we now derive

$$\inf_{v \in S_R \cap V_n(y)^\perp} \psi(v) - \psi(y) > 0$$

and therefore

$$c_n(R) - \psi(y) > 0,$$

as claimed.

Corollary 2.7 *Suppose hypotheses (H1) – (H3) and (CC) are satisfied. Let* $n \in \mathbb{N}$, $R > 0$, $\tilde{R} > 2c_n(R)$ *and consider* $y \in \partial T(R, \tilde{R})$. *If* $y \in V_n(y)$, *then:*

(a) $y \in S_R$.

(b) If $\psi(y) = c_n(R)$, *then* (λ, y) *is a solution of problem (1.1), (1.2) for* $\lambda = \mu_n(y)$.

Proof: Since y satisfies the assumptions of Lemma 2.6, there holds $\psi(y) \leq c_n(R)$. Hence, using (2.22) we observe that $\|y\|_X < \tilde{R}$, and therefore there must hold $\|y\| = R$. This shows (a).
If moreover $\psi(y) = c_n(R)$, then Lemma 2.6(b) implies that $\mu_n(y)$ *has to* be an eigenvalue of $A(y)$. Thus the inequalities (2.23) and (2.24) hold, whereas we know from (a) that $\|y\| = R$. Thus, if $\psi(y) = c_n$, there actually must hold $\rho(y) = \mu_n(y)$, which is possible only if y is an eigenvector of $A(y)$ with eigenvalue $\mu_n(y)$.

Corollary 2.8 *Suppose hypotheses (H1) – (H3), (CC) are satisfied. Moreover consider* $n \in \mathbb{N}$, $R > 0$ *and* $\tilde{R} > 2c_n(R)$. *Then the set* K *equals* $\tilde{K}$, *where*

$$\tilde{K} := \{y \in \partial T(R, \tilde{R}) |\; y \in V_n(y)\}.$$

Proof: From corollary (2.7) we directly deduce $\tilde{K} \subset K$. For the opposite inclusion pick $y \in K$. Then $\psi(y) \leq c_n(R)$ by lemma (2.6). Combining this with (2.22), we get $\|y\|_X < \tilde{R}$, hence y has to be an element of $\tilde{K}$.

Our main theorem will now follow simply by combining the above results with a topological lemma (Prop. 2.1) from [10].

Theorem 2.9 *Suppose (H1) – (H3) and (CC) are satisfied. Moreover consider* $n \in \mathbb{N}$, $R > 0$ *and* $\tilde{R} > 2c_n(R)$ *such that* $(EC)_{n,D}$ *holds for* $D := T(R, \tilde{R})$. *Then condition (CP) from Section 1 holds true for these values of* R *and* n.

Proof: We apply proposition 2.1 from [10], using the notation from there without further explaination. We set $S := \partial T(R, \tilde{R})$. Then clearly $S \in \Sigma$, and S is bounded even in X (as well as in $\mathcal{H}$). Moreover, S is homeomorphic to the unit sphere $S_1 = \{y \in X |\; \|y\|_X = 1\}$ by radial projection, since S is the unit sphere with respect to an equivalent norm on X. We now define a map $H : [0, 1] \times S \to \mathcal{L}(X)$ by

$$H(t, y) := P_n(ty),$$

where P_n is defined as in proposition 2.5. This proposition ensures that H is continuous, and that the assumptions (iii) and (iv) of [10, proposition 2.1] are satisfied. Furthermore, condition (i) is a consequence of the fact that $A(y)$ is an even function of $y \in X$ by virtue of (H3). Finally $H(0,\cdot) \equiv P_n(0)$ is constant on S, thus (ii) holds as well. Application of proposition 2.1 now implies that the set $\tilde{K}$ (as defined in Corollary 2.8) has the following properties:
$\tilde{K} \in \Sigma$, $\tilde{K}$ is compact and $\gamma(K) = n$. But actually $K = \tilde{K}$, so the same holds for K. In particular, ψ attains its maximum on K, and by definition of $c_n(R)$ there holds $\max\limits_{u \in K} \psi(u) \geq c_n(R)$. But actually equality holds by virtue of lemma (2.6). Finally, corollary 2.7(b) ensures that every $y \in \psi^{-1}(c_n(R))$ is a solution of equation (1.1) with eigenvalue $\lambda = \mu_n(y)$.
Hence, (CP) holds true and the theorem is proved.

So far we always assumed (2.1) for the operator A_0, but, as announced, we now consider the case that A_0 is only assumed to be bounded from below. Thus let $m \in \mathbb{R}$ satisfy

$$(A_0 v, v) \geq -m\|v\|^2 \qquad (v \in \mathcal{D}(A_0)). \tag{2.25}$$

By virtue of (H1) and (H2) we may clearly define the form sum $A(y) := A_0 + \tilde{B}(y)$ for each $y \in X$. However, in order to apply Theorem 2.9, we have to pass to the operators $\tilde{A}(y) := A(y) + (m+1)I$ and the functional $\tilde{\psi}$ defined by

$$\tilde{\psi}(y) := \psi(y) + (m+1)\|v\|^2.$$

Since the transformation $\psi \mapsto \tilde{\psi}$ will change the Ljusternik-Schnirelman levels as $c_n(R) \mapsto c_n(R) + (m+1)R^2$, this application directly leads to

Theorem 2.10 *Suppose that instead of (2.1) there just holds (2.25), and suppose that (H1) – (H3) and (CC) are satisfied. Moreover consider $n \in \mathbb{N}$, $R > 0$ and $\tilde{R} > 2[c_n(R) + (m+1)R^2]$ such that $(EC)_{n,D}$ holds for $D := T(R, \tilde{R})$. Then condition (CP) from Section 1 holds true for these values of R and n.*

Theorem 2.10 entails a remarkable further observation concerning the multiplicity of the values $c_n(R)$:

Corollary 2.11 *Suppose that (H1)-(H4) and (CC) are satisfied. Furthermore consider $n, p \in \mathbb{N}$, $R > 0$, $\tilde{R} > 2[c_{n+p}(R) + (m+1)R^2]$ such that the conditions $(EC)_{n,D}$ as well as $(EC)_{n+p,D}$ hold for $D = T(R, \tilde{R})$. Then*

$$c_n(R) < c_{n+p}(R) \tag{2.26}$$

Proof: From Theorem 2.10 we deduce that the condition (CP) is satisfied for both of the two sets

$$K^n := \{y \in S_R |\; y \in V_n(y)\}$$

and

$$K^{n+p} := \{y \in S_R |\, y \in V_{n+p}(y)\}.$$

To derive a contradiction, suppose that $c_n = c_{n+1} = ... = c_{n+p}$. Then we could choose an $y \in K^n \subset K^{n+p}$ such that

$$\psi(y) = c_n = c_{n+p}.$$

Consequently condition (CP) would imply that

$$\lambda_n(y)R^2 = (A(y)y, y) = \lambda_{n+p}(y)R^2,$$

But, noting that $\|y\|_X \leq \tilde{R}$, this contradicts our assumption $(EC)_{n,D}$.
Therefore $c_n(R) < c_{n+p}(R)$, as claimed.

We emphasize that this is a new way to exclude higher multiplicity of the c_n in a quite abstract context, using only spectral properties which are well known in many cases. In case of differential and integral equations, results on the multiplicity of the c_n have been obtained in many situations (see e.g. [5],[8],[10],[11]). Most of these results are based on examining the topology of the solution set of (1.1), following an idea from [5]. However, at least in our application to radially symmetric Schrödinger equations on $\mathbb{R}^N$(see section 3) it is not obvious how this eximination can be done. Corollary 2.11 does the job for this case, and it can also be applied to the equations considered in [10].

3 Radially Symmetric Schrödinger Equations

We consider the radially symmetric semilinear elliptic equation

$$-\Delta u + q(|x|)u + f(|x|, u^2)u = \lambda u, \qquad x \in \mathbb{R}^N, \tag{3.1}$$

together with the isoperimetric constraint

$$\int_{\mathbb{R}^N} u(x)^2 \, dx = R^2 \tag{3.2}$$

for real valued functions $u \in W^{1,2}(\mathbb{R}^N)$ with a given $R > 0$. (Here and in the sequel $|\cdot|$ denotes the euklidian norm on $\mathbb{R}^N$, and we will sometimes write r for $|x|$). We assume $N \geq 2$ and denote

$$2^* := \begin{cases} \frac{2N}{N-2} & \text{for} \quad N \geq 3 \\ \infty & \text{for} \quad N = 2 \end{cases}$$

the critical Sobolev exponent. The data functions $q :]0, \infty[\to \mathbb{R}$ and $f :]0, \infty[\times [0, \infty[$ satisfy the following assumptions:

(I1) q is a continuous function on $]0, \infty[$ satisfying

$$\lim_{r \to \infty} q(r) = 0, \tag{3.3}$$

but also

$$0 > \limsup_{r \to \infty} q(r) r^{\gamma} \geq -\infty \tag{3.4}$$

for some $\gamma \in]0, 2[$ as well as

$$\int_0^{\varepsilon} r|q(r)|\, dr < \infty \qquad \text{in case } N \geq 3 \text{ resp.}$$

$$\int_0^{\varepsilon} r|ln(r)q(r)|\, dr < \infty \qquad \text{in case } N = 2.$$

(I2) f is continuous, $f(r, 0) \equiv 0$,

(I3) There are constants $c > 0$, $\alpha \in \mathbb{R}$, $\beta > 0$ such that

$$|f(r, s)| \leq c r^{\alpha} s^{\beta} \text{ for } 0 < r < \infty$$

and $\alpha - (N-2)\beta > -2$ as well as $\alpha - (N-1)\beta < 0$,

(M) For every $r \in]0, \infty[$, $f(r, \cdot)$ is monotonically nondecreasing on $[0, \infty[$.

Condition (I1) guarantees that $q \in K_N$(the Kato class, see e.g. [18]), hence q is $-\Delta$-form bounded with relative bound zero (cf. [2, Theorem 4.7]). Therefore, by the KLMN-Theorem(see e.g. [16]), the form sum $-\Delta + q$ is a well defined selfadjoint and semi-bounded operator in $L^2(\mathbb{R}^N)$ with form domain $W^{1,2}(\mathbb{R}^N)$. Moreover, the essential spectrum of this operator is $[0, \infty)$ as a consequence of (3.3), cf. [17].

Let us cast this problem in the abstract setting of section 2: In the following $\mathcal{H}$ resp. X, $\mathcal{D}(A_0)$ denote the closed subspaces consisting of the radially symmetric functions in $L^2(\mathbb{R}^N)$ resp. $W^{1,2}(\mathbb{R}^N)$, $\mathcal{D}(-\Delta + q)$. The operator $A_0 : D(A_0) \subset \mathcal{H} \to \mathcal{H}$ is defined as the restriction of $-\Delta + q$ to $\mathcal{D}(A_0)$(Note that $-\Delta + q$ commutes with rotations, therefore A_0 is well defined and selfadjoint). Furthermore, A_0 is bounded from below since $-\Delta + q$ is. Let m be a lower bound for A_0, i.e. we have

$$(A_0 v, v) \geq m \|v\|^2,$$

where $(\cdot, \cdot)$ resp.$\|\cdot\|$ denote the scalar product resp. the norm in $\mathcal{H} \subset L^2(\mathbb{R}^N)$. Let also $\|\cdot\|_X$ denote the norm in $X \subset W^{1,2}(\mathbb{R}^N)$, so that we retain the notation from section 2 .

Note that the growth conditions for f do *not* guarantee that for every $y \in X$ the domain $\mathcal{D}(\tilde{B}(y)) \subset H$ of the multiplication operator

$$\tilde{B}(y) : \quad u \mapsto f(r, y^2) u \tag{3.5}$$

contains $\mathcal{D}(A_0)$. But taking $u \in X$, the map (3.5) defines, for each $y \in X$, a canonical element of X^*, and this map satisfies our abstract conditions. Indeed we have:

Lemma 3.1

(a) There is a completely continuous map $B : X \to \mathcal{L}(X, X^*)$ given by

$$\langle B(y)v, w\rangle := \int_{\mathbb{R}^N} f(|x|, y^2)v(x)w(x)dx$$

such that $B(y)$ is a compact linear operator for each $y \in X$.

(b) The functional $\varphi : X \to \mathbb{R}$ defined by

$$\varphi(u) := \frac{1}{2}\int_{\mathbb{R}^N}\int_0^{u^2(x)} f(|x|, s)\, ds\, dx \qquad (u \in X) \tag{3.6}$$

is continuous. Moreover, φ and B satisfy (CC).

In fact one can prove that φ is even a C^1-functional with derivative given by $\partial\varphi(u) = B(u)u$ for $u \in X$, but we do not need this here. As a consequence of Lemma 3.1, we may introduce $A(y)$, $\mu_n(y)$ and $V_n(y)$ for $y \in X$ as well as ψ and $c_n(R)$ for $R > 0$ in the same way as done in section II. Moreover, (H1) and (H2) are satisfied, while (H3) clearly follows from (I2) and the definition of B. For the moment let us take Lemma 3.1 for granted and turn to the main result of this section:

Theorem 3.2 *Suppose assumptions (I1) – (I3) and (M) are satisfied. Consider $n \in \mathbb{N}$ and set*

$$K_R := \{y \in S_R | y \in V_n(y)\}$$

for $R > 0$. Then we have

(a) For $R > 0$ sufficiently small there holds

i) K_R is compact and symmetric, and $\gamma(K_R) = n$. In particular, $K \neq \emptyset$.

ii) $c_n = \max_{y \in K_R} \psi(y)$.

iii) Every $y \in K_R \cap \psi^{-1}(c_n(R))$ is a strong solution of (3.1) – (3.2) belonging to $C(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\})$ and having precisely n nodal surfaces in $\mathbb{R}^N \setminus \{0\}$.

iv) $c_n(R) < c_{n+1}(R)$.

(b) Assume in addition to (I3) that $\alpha - (N-1)\beta < -\gamma$. Then the assertions i)-iv) remain valid for arbitrary $R > 0$.

Remarks. (a) It is elucidating to consider the autonomous case (α =0) in particular. Because the critical Sobelev exponent corresponds to $\beta = 2/(N-2)$, we see that any superlinear and subcritical growth is allowed to ensure (I3), and therefore Theorem 3.2 establishes the stated properties of K_R for sufficiently small R. To obtain the more interesting *global* result, i.e. the case of arbitrary $R > 0$, one only needs the weak further restriction $\beta > \dfrac{\gamma}{N-1}$, i.e. the order of the nonlinearity near zero must be higher than $1 + \dfrac{2\gamma}{(N-1)}$. It is not clear wheather this restriction can be removed.

(b) For the solutions to which we refer in (iii), we cannot expect more regularity at the origin. Actually that is not even ensured in the linear case (note that $f(\cdot,\cdot) \equiv 0$ is a possible choice under our assumptions). To give an easy example, the (up to a constant) unique eigenfunction u of

$$(-\Delta - \frac{2}{r})u = \lambda u$$

associated to the lowest eigenvalue $\lambda_{min} = -1$ is given by $u(x) = e^{-|x|}$. Obviously u is not differentiable in $x = 0$.

The remainder of the section is devoted to the proof of Theorem 3.2, and from now on we always assume (I1)–(I3) and (M) to be satisfied. Moreover, in view of (I3) we can fix numbers $p, q \in (2, 2^*)$, $0 < \varepsilon_0 < 1$ such that

$$\alpha - (N - 2 + \varepsilon_0)\beta > \frac{N - 2 + \varepsilon_0}{2}(2 - p) > -2 + \varepsilon_0 \tag{3.7}$$

as well as

$$\alpha - (N-1)\beta < \frac{N-1}{2}(2-q) \qquad \text{and} \qquad q - 2 \le 2\beta. \tag{3.8}$$

Now we are prepared to begin the *Proof of Lemma 3.1:*

(a) We will show the assertion by giving an appropriate factorization for B. We therefore introduce for $1 \le s \le \infty$ the Banach spaces

$$V_s := L^s(\mathcal{C}) \quad \text{and } W_s := L^s(\mathbb{R}^N \setminus \mathcal{C}),$$

$\mathcal{C}$ denoting the unit ball in $\mathbb{R}^N$. In particular, since $2 < p, q < 2^*$, we conclude that the restriction maps

$$i : X \to V_p \qquad u \overset{i}{\mapsto} u|_{\mathcal{C}}$$

and

$$j : X \to W_p \qquad u \overset{j}{\mapsto} u|_{\mathbb{R}^N \setminus \mathcal{C}}$$

are compact(cf. e.g. [24]). Now consider the following factorisation for B:

$$X \overset{(f_*^1,f_*^2)}{\longrightarrow} V_{\frac{p}{p-2}} \times W_{\frac{q}{q-2}} \overset{b}{\longrightarrow} \mathcal{L}(V_p, V_{p'}) \times \mathcal{L}(W_q, W_{q'}) \overset{g}{\longrightarrow} \mathcal{L}(X, X^*).$$

Here for $u \in X$ we set

$$f_*^1(u) := f(\cdot, u^2(\cdot))|_{\mathcal{C}}, \qquad f_*^2(u) := f(\cdot, u^2(\cdot))|_{\mathbb{R}^N \setminus \mathcal{C}}$$

and the map b is defined by $b = (b_1, b_2)$ with

$$\begin{aligned} b_1(v,w)u_1 &:= vu_1, \\ b_2(v,w)u_2 &:= wu_2 \end{aligned}$$

for $(v,w) \in V_{\frac{p}{p-2}} \times W_{\frac{q}{q-2}}$ and $(u_1, u_2) \in V_p \times W_q$. Finally g maps a pair of linear operators

$$h = (h_1, h_2) \in \mathcal{L}(V_p, V_{p'}) \times \mathcal{L}(W_q, W_{q'}) \cong \mathcal{L}(V_p, V_p^*) \times \mathcal{L}(W_q, W_q^*)$$

to

$$g(h) = i^* h_1 i + j^* h_2 j \in \mathcal{L}(X, X^*).$$

Obviously the maps b and g are continuous, and the range of g consists only of *compact* operators. Thus it now suffices to prove that the nonlinear operators f_*^1 resp. f_*^2 are well-defined (i.e. they map into the right spaces), and that they are compact and continuous. So let us first consider f_*^1:
Using (I3), we have

$$\begin{aligned} |f_*^1(u)(x)|^{\frac{p}{p-2}} &= |f(r, u^2(x))|^{\frac{p}{p-2}} \\ &\leq cr^{\alpha\frac{p}{p-2}} |u(x)|^{2\beta\frac{p}{p-2}} \end{aligned} \tag{3.9}$$

We now have to treat the two cases $p-2 > 2\beta$ and $p-2 \leq 2\beta$ seperately. In the first case Hölder's inequality leads to

$$\int_{\mathcal{C}} |f_*^1(u)(x)|^{\frac{p}{p-2}}\, dx \;\leq\; c\left(\int_{\mathcal{C}} r^{\frac{\alpha p}{p-2-2\beta}}\right)^{\frac{\alpha p}{p-2}} \|u\|_{V_p}^{2\beta\frac{p}{p-2}}, \tag{3.10}$$

while (3.7) ensures that for $\alpha < 0$ there holds $\frac{\alpha p}{p-2-2\beta} > -\frac{p}{2}(N-2+\varepsilon_0) > -N$, hence the integral on the right hand side of (3.10) exists. Hence f_*^1 maps X into $V_{\frac{p}{p-2}}$.
Next we show that f_*^1 is strongly continuous (i.e. if $u_n \rightharpoonup u$ in X, then $f_*^1(u_n) \to f_*^1(u)$ in $V_{\frac{p}{p-2}}$). This implies both the continuity and the compactness of f_*^1, since in the Hilbert space X every bounded sequence contains a weakly convergent subsequence. So consider a sequence $(u_n)_n \subset X$ with $u_n \rightharpoonup u \in X$. We assume by contradiction that

$$\|f_*^1(u_{n_k}) - f_*^1(u)\|_{V_{\frac{p}{p-2}}} \geq \delta > 0 \tag{3.11}$$

for a subsequence (u_{n_k}) of (u_n). Since $u_{n_k}|_{\mathcal{C}} \to u|_{\mathcal{C}}$ in V_p, we find that (passing again to a subsequence if necessary) there exists $g \in V_p$ such that $|u_{n_k}(x)| \leq g(x)$ and $u_{n_k}(x) \to u(x)$ almost everywhere on $\mathcal{C}$ (see i.e. [24, Lemma A.1]). Since f is continuous, we conclude

$$f_*^1(u_{n_k})(x) \to f_*^1(u)(x) \qquad \text{a.e. on } \mathcal{C}.$$

Moreover, using (3.9), we obtain

$$|f_*^1(u_{n_k})(x)|^{\frac{p}{p-2}} \leq c r^{\alpha \frac{p}{p-2}} g(x)^{2\beta \frac{p}{p-2}} \qquad \text{a.e. on } \mathcal{C}.$$

where the integral over the ride hand side exists again by Hölder's inequality. Hence, by Lebesgue's Theorem,

$$\lim_{k\to\infty} \|f_*^1(u_{n_k}) - f_*^1(u)\|_{V_{\frac{p}{p-2}}} = 0,$$

which contradicts (3.11). Thus we have shown that f_*^1 is strongly continuous in case that $p-2 > 2\beta$.

To treat the case $p - 2 \leq 2\beta$ we recall that by a well known estimate concerning radially symmetric functions (cf. [14, p. 317]) there is a constant $K > 0$ such that

$$|u(x)| \leq K \|u\|_X \, r^{-\frac{N-2+\varepsilon_0}{2}} \qquad (r = |x|) \tag{3.12}$$

for $u \in X$, $x \in \mathbb{R}^N$. Combining this with (3.9) we get

$$\begin{aligned} |f_*^1(u)(x)|^{\frac{p}{p-2}} &\leq c r^{\alpha \frac{p}{p-2}} |u(x)|^p |u(x)|^{2\beta \frac{p}{p-2} - p} \\ &\leq c \left(K \|u\|_X\right)^{\frac{p}{p-2}(2\beta - (p-2))} |u(x)|^p \tau(r) \end{aligned} \tag{3.13}$$

with

$$\tau(r) = r^{\frac{p}{p-2}[\alpha - \frac{N-2+\varepsilon_0}{2}(2\beta - (p-2))]} \leq 1$$

for $r = |x| \leq 1$ in view of (3.7). Again we conclude that f_*^1 maps X into $V_{\frac{p}{p-2}}$.

To show that f_*^1 is strongly continuous we again consider sequences $(u_n)_n$, $(u_{n_k})_k \subset X$ and $g \in V_p$ as above. Hence, by the same arguments as before, we only need to ensure that the functions $|f_*^1(u_{n_k})(\cdot)|^{\frac{p}{p-2}}$ are dominated on $\mathcal{C}$ by an integrable funtion. Indeed this is true, since in view of (3.13) we conclude

$$|f_*^1(u_{n_k})(x)|^{\frac{p}{p-2}} \leq \text{const } g(x)^p \qquad \text{a.e. on } \mathcal{C}.$$

(Note that the sequence $(u_n)_n$ is bounded in X).

Analogous arguments show that f_*^2 is well defined and strongly continuous, but since $q - 2 \leq 2\beta$ by assumption, we do not need to distinguish cases now. Instead of (3.12) one now uses the estimate

$$|u(x)| \leq \tilde{K} \|u\|_X r^{-\frac{N-1}{2}} \qquad (r = |x|) \tag{3.14}$$

for $u \in X$, $x \in \mathbb{R}^N$ (cf. again [14, p. 317]). This leads to

$$|f_*^2(u)(x)|^{\frac{q}{q-2}} \;\; \leq \;\; c\left(\tilde{K}\|u\|_X\right)^{\frac{q}{q-2}(2\beta-(q-2))} \tilde{\tau}(r)|u(x)|^q$$

with

$$\tilde{\tau}(r) = r^{\frac{q}{q-2}[\alpha - \frac{N-1}{2}(2\beta-(q-2))]} \leq 1$$

for $r = |x| \geq 1$ in view of (3.8). Therefore $f_*^2(u) \in V_{\frac{q}{q-2}}$, and the strong continuity of f_*^2 is now derived just as for f_*^1. This completes the proof of (a).

(b) Let $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ denote the continuous function defined by

$$g(x,t) := \int_0^{t^2} f(|x|, s)\, ds.$$

We show that we have substitution operators $g_*^1 : V_p \to V_1$ and $g_*^2 : W_q \to W_1$ both defined pointwise by

$$(g_*^i(u))(x) := g(x, u(x)).$$

So pick $u \in V_p$ first, and note that (I3) implies

$$|g(x,u(x))| \leq \text{const } r^\alpha |u(x)|^{2\beta+2}$$

Again we treat the cases $p-2 > 2\beta$ and $p-2 \leq 2\beta$ seperately: For $p-2 > 2\beta$ Hölder's inequality yields

$$\int_{\mathcal{C}} |g(x,u(x))|\, dx \leq \text{const } \left(\int_{\mathcal{C}} r^{\frac{\alpha p}{p-2-2\beta}}\, dx\right)^{\alpha} \|u\|_{V_p}^{2\beta+2}$$

since we have already ensured that the integral on the right hand side exists. For $p-2 \leq 2\beta$ by (3.14) we get

$$\begin{aligned} |g(x,u(x))| &\leq \text{const } r^\alpha |u(x)|^{2\beta+2} \\ &\leq \text{const } r^\alpha |u(x)|^p |u(x)|^{2\beta-(p-2)} \\ &\leq \text{const } r^{\alpha - \frac{N-2+\varepsilon_0}{2}(2\beta-(p-2))} |u(x)|^p \\ &\leq \text{const } |u(x)|^p \end{aligned}$$

So in either case $g_*^1(u) \in V_1$. Recalling that $q-2 \leq 2\beta$ by assumption, one derives $g_*^2(u) \in W_1$ for $u \in W_q$ in an analogous way. Since the g_*^i are Nemytzkii operators, they are continuous. But since for $u \in X$ we have

$$\varphi(u) = \frac{1}{2}\left(\int_{\mathcal{C}} g(x,u(x))\, dx + \int_{\mathbb{R}^N \setminus \mathcal{C}} g(x,u(x))\, dx\right),$$

it follows that φ is continuous as well.
It remains to prove (CC) for B and φ. To this end, note that for arbitrary $r \in]0,\infty[$ the function

$$F(r,\cdot) := \int_0^{(\cdot)} f(r,s)ds$$

is convex, since f satisfies (M). In particular, for $y, v \in X$ and $x \in \mathbb{R}^N \setminus \{0\}$ there holds

$$\begin{aligned} F(|x|, v^2(x)) - F(|x|, y^2(x)) &\geq \partial_2 F(|x|, y^2(x))(v^2(x) - y^2(x)) \\ &= f(|x|, y^2(x))(v^2(x) - y^2(x)). \end{aligned}$$

Recalling that

$$2\varphi(u) = \int_{\mathbb{R}^N} F(|x|, u^2(x))\, dx, \qquad (u \in X)$$

condition (CC) follows by integration.

Finally we obtain a quite general result concerning our condition $(EC)_{n,D}$.

Lemma 3.3

(a) Let $n \in \mathbb{N}$ be given. Then condition $(EC)_{n,D_{\tilde{R}}}$ is satisfied for $\tilde{R} > 0$ sufficiently small.

(b) Assume in addition to (I3) that $\alpha - (N-1)\beta < -\gamma$. Then condition $(EC)_{n,D}$ is satisfied for every $n \in \mathbb{N}$ and any *bounded subset $D \subset X$.*

Proof: We start with the proof of (b): Since D is bounded in X, we can (recalling (3.14)) choose $M > 0$ such that for all $y \in D$ there holds

$$|y(x)| \leq M r^{-(N-1)\beta} \qquad (x \in \mathbb{R}^N).$$

In view of 3.4, pick $K, R_0 > 0$ such that $q(r) \leq -\dfrac{K}{r^\gamma}$ for $r \geq R_0$. Furthermore pick a nonnegative radially symmetric function $\psi \in C_0^\infty(\mathbb{R}^N)$ with support in $\{x | 1 < |x| < 2\}$ and $\|\psi\| = 1$. For $R > R_0$, let $\psi_R(x) = R^{-N/2}\psi(x/R)$, thus $\|\psi_R\| = 1$ and

$$\operatorname{supp} \psi_R \subset \{x |\, R < |x| < 2R\}.$$

Clearly $\psi_R \in \mathcal{D}(A(y))$ for every $y \in X$, and for $y \in D$ we conclude

$$\begin{aligned} (A(y)\psi_R, \psi_R) &= (-\Delta\psi_R, \psi_R) + (q\psi_R, \psi_R) + \langle B(y)\psi_R, \psi_R\rangle \\ &\leq (-\Delta\psi_R, \psi_R) + ((-Kr^{-\gamma} + cMr^{\alpha-(N-1)\beta})\psi_R, \psi_R) \end{aligned}$$

Now since $\alpha - (N-1)\beta < -\gamma$ by assumption, there is an $R_1 > R_0$ such that

$$-Kr^{-\gamma} + cMr^{\alpha-(N-1)\beta} < -\frac{K}{2}r^{-\gamma}$$

for $r \geq R_1$, therefore

$$\begin{aligned} (A(y)\psi_R, \psi_R) &\leq (-\Delta\psi_R, \psi_R) - \frac{K}{2}(r^{-\gamma}\psi_R, \psi_R) && (3.15)\\ &= R^{-2}(-\Delta\psi, \psi) - \frac{K}{2}R^{-\gamma}(r^{-\gamma}\psi, \psi) && (3.16) \end{aligned}$$

for $R > R_1$. Since $\gamma < 2$, the last expression, viewed as a function of R, tends to zero monotonically from below if we consider R large enough. In particular we can find $R_2 > R_1$, $\varepsilon > 0$ so that $(A(y)\psi_R, \psi_R) < -\varepsilon$ for $R_2 < R \leq 2^n R_2$, $y \in D$. Let $u_k := \psi_{2^k R_2}$, $k = 1, \ldots, n$ and $W_n := span\{u_1, ..., u_n\}$. Since the u_k are orthonormal and $(A(y)u_j, u_k) = 0$ for $j \neq k$, there holds

$$\sup_{u \in W_n, \|u\|=1} (A(y)u, u) \leq -\varepsilon$$

for all $y \in D$. Thus,

$$\sup_{y \in D} \mu_n(y) \leq -\varepsilon < 0 \leq \inf \sigma_{ess}(A_0),$$

which proves (2.15) since $\mu_\infty = \inf \sigma_{ess}(A_0)$. We still need to ensure that $\mu_n(y) < \mu_{n+1}(y)$ for every $y \in D$, $n \in \mathbb{N}$. But since the equation $A(y)y = \mu y$ can be rewritten as a singular boundary value problem of Sturm-Liouville type, $A(y)$ has only simple eigenvalues (cf. [23, p. 257]).

(a): It is not difficult to see that analogous but simpler arguments as above show that for the special choice $y = 0$ the relations

$$\mu_1(0) < \mu_2(0) < ... < 0 = \inf \sigma_{ess}(A_0)$$

hold. Set $\eta := \mu_{n+1}(A_0) - \mu_n(A_0)$, and let V denote the direct sum of the one dimensional eigenspaces corresponding to $\mu_1(A_0), .., \mu_n(A_0)$. Then there is a constant $C > 0$ such that for $y \in X$, $v \in V$

$$\begin{aligned} |(A(y)v, v) - (A_0 v, v)| &= |\langle B(y)v, v\rangle| \\ &\leq \|B(y)\|_{\mathcal{L}(X,X^*)}\|v\|_X^2 \\ &\leq C\|B(y)\|_{\mathcal{L}(X,X^*)}\|v\|^2, \end{aligned}$$

since V is finite dimensional. Recalling that B is continuous in $y = 0$, we can choose $\tilde{R} > 0$ such that $\|B(y)\|_{\mathcal{L}(X,X^*)} \leq \frac{\eta}{2C}$ for $y \in D_{\tilde{R}}$. Thus

$$|(A(y)v, v) - (A_0 v, v)| \leq \frac{\eta}{2}\|v\|^2$$

for $y \in D_{\tilde{R}}$, $v \in V$, and therefore $\mu_n(y) < \mu_{n+1}(0) < 0$. This shows (2.15). On the other hand, since $\langle B(y)v, v\rangle \geq 0$ for *all* $y, v \in X$ by defintion, there holds $\mu_{n+1}(y) \geq \mu_{n+1}(0)$. Hence $\mu_{n+1}(y) > \mu_n(y)$ for $y \in D_{\tilde{R}}$, and $(EC)_{n,D_{\tilde{R}}}$ is valid.

Lemma 3.4 *$c_n(R) \to 0$ for $R \to 0$.*

Proof: Defining the n-dimensional vector space V as in the proof of Lemma 3.4(a), we have for $y \in V$:

$$\begin{aligned}\psi(y) &= \varphi(y) + \frac{1}{2}\|y\|_X^2 \\ &\leq \frac{1}{2}\langle B(y)y, y\rangle + \frac{1}{2}\|y\|_X^2 \\ &\leq C(\|B(y)\|_{\mathcal{L}(X,X^*)} + 1)\|y\|^2,\end{aligned}$$

Now

$$\begin{aligned}\lim_{R\to 0} \sup_{y\in V,\, \|y\|=R} \|B(y)\|_{\mathcal{L}(X,X^*)} &= \lim_{R\to 0} \sup_{y\in V,\, \|y\|_X=R} \|B(y)\|_{\mathcal{L}(X,X^*)} \\ &= 0\end{aligned}$$

because B is continuous in $y = 0$, and therefore

$$\lim_{R\to 0} \sup_{y\in V,\, \|y\|=R} \psi(y) = 0.$$

Since $\{y \in V,\ \|y\| = R\} \in \Sigma_n(R)$, this implies $c_n(R) \to 0$, as claimed.

Summarizing the preceeding lemmas, the *Proof of Theorem 3.2* now easily follows:
(a) Let $n \in \mathbb{N}$ be given. In view of Lemma 3.3(a) and Lemma 3.4 we observe that for suffiently small $R > 0$ there is a $\tilde{R} > 2[c_n(R) + (m+1)R^2]$ such that $(EC)_{n,D_{\tilde{R}}}$ is valid. Since $T(R, \tilde{R}) \subset D_{\tilde{R}}$, the hypothesis of Theorem 2.10 are satisfied with the parameters $n, R, \tilde{R}$. Thus condition (CP) holds, which implies our assertions (i) and (ii).
As a further consequence, every $y \in K \cap \psi^{-1}(c_n)$ is a strong solution of (3.1)-(3.2) corresponding to $\lambda = \mu_n(y)$. Fixing y with these properties, we find that $y \in C^2(\mathbb{R}^N \setminus \{0\})$, since y, viewed as a function of $r = |x|$, solves an ordinary differential equation of second order with continuous coefficients on $]0, \infty[$.
To show the continuity of y in the origin, consider again the unit ball $\mathcal{C} \subset \mathbb{R}^N$. By (3.13) we see that $f_*^1(y) \in L^{\frac{p}{p-2}}(\mathcal{C})$, where $\frac{p}{p-2} > \frac{n}{2}$ since $p < 2^*$. We conclude that

$f(|\cdot|, y^2(\cdot)) \in K_N^{loc}$, hence also the function $V := q + f(|\cdot|, y^2(\cdot))$. Since y is a distributional solution of

$$(-\Delta + V - \mu_n(y))y = 0$$

y is continuous by virtue of [2, Theorem 1.5], for instance. It remains to show that y has precisely n nodal surfaces in $\mathbb{R}^N \setminus \{0\}$. Using again the reformulation of the equation $A(y)y = \mu_n(y)y$ as a selfadjoint singular Sturm-Liouville problem on $]0, \infty[$, we find that $y = y(r)$ must have exactly $n-1$ zeros in $]0, \infty[$ by classical characterizations of eigenfunctions corresponding to the $n-th$ eigenvalue (see e.g. [7, p.1480]). Finally, (iv) arises from Corollary 2.11 together with Lemma 3.3(a).

(b) Under the additional assumption $\alpha - (N-1)\beta < -\gamma$ Lemma 3.3(b) ensures the hypothesis of Theorem 2.10 is satisfied for *every* given $n \in \mathbb{N}$, $R > 0$ with a suitable $\tilde{R} > 0$. Therefore condition (CP) is established, and (i),(ii) hold true. (iii) follows as in the proof of part (a), and finally (iv) is a consequence of Corollary 2.11 together with Lemma 3.3(b).

4 Sublinear elliptic equations on other radially symmetric domains

In this section we give a brief sketch of the results our theory yields for the boundary value problem

$$-\Delta u + q(|x|)u + f(|x|, u^2)u = \lambda u, \qquad x \in \Omega, \tag{4.1}$$

$$u|_{\partial\Omega} = 0 \tag{4.2}$$

with the constraint

$$\int_\Omega u(x)^2 \, dx = R^2, \tag{4.3}$$

where again $u \in W^{1,2}(\Omega)$ as well as the data functions q and f are supposed real valued. We consider connected radially symmetric domains $\Omega \neq \mathbb{R}^N$, so that there are three different cases:

(i) A ball $\Omega := \{x \in \mathbb{R}^N \,|\, |x| < r\}$ with radius $r < \infty$.

(ii) An annulus $\Omega := \{x \in \mathbb{R}^N \,|\, r_1 < |x| < r_2\}$ with $0 < r_1 < r_2 < \infty$.

(iii) An exterior domain $\Omega := \{x \in \mathbb{R}^N \,|\, |x| > r\}$ with $0 < r$.

We might expect better results than in the previous section, since we have removed singularities from the problem. Indeed this is true, as will turn out further below. We introduce the following assumptions:

($\mathrm{I1}_\Omega$) $q(|\cdot|)$ is continuous on $\Omega \setminus \{0\}$, and

$$\text{in case (i):} \quad \begin{cases} \int_0^\varepsilon r|q(r)| < \infty & \text{if } N \geq 3 \text{ resp.} \\ \int_0^\varepsilon r|ln(r)q(r)| < \infty & \text{if } N = 2. \end{cases}$$

$$\text{whereas in case (iii):} \quad \begin{cases} \lim\limits_{r\to\infty} q(r) = 0 & \text{and} \\ -\infty \leq \limsup\limits_{r\to\infty} q(r)r^\gamma < 0 & \text{for some } \gamma \in (0,2) \end{cases}$$

($\mathrm{I2}_\Omega$) f is continuous, $f(r,0) \equiv 0$,

($\mathrm{I3}_\Omega$) There are constants $c > 0$, $\alpha \in \mathbb{R}$, $\beta > 0$ such that

$$|f(r,s)| \leq cr^\alpha s^\beta \quad \text{for} \quad 0 < r < \infty$$

and

$$\begin{aligned} \alpha - (N-2)\beta &> -2 \qquad \text{in case (i)} \\ \alpha - (N-2)\beta &< -\gamma \qquad \text{in case (iii)} \end{aligned}$$

(M_Ω) For every $x \in \Omega$, $f(|x|,\cdot)$ is monotonically nondecreasing on $[0,\infty[$.

Again condition ($\mathrm{I1}_\Omega$) ensures that q is $-\triangle$-form bounded with relative bound zero, so that the form sum $-\triangle + q$ exists as a selfadjoint semi-bounded operator in $L^2(\Omega)$ with form domain $W_0^{1,2}(\Omega)$. In case (i) this follows essentially as for $\Omega = \mathbb{R}^N$(see also [1, Lemma 0.3] or [17, Lemma 7.5]), whereas in cases (ii) and (iii) q is even $-\triangle$-compact so that there exists the operator sum $-\triangle+q$ with domain $W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$.

To cast this problem into the framework of section II, we proceed just as in section III. So let $\mathcal{H}$ resp. X, $\mathcal{D}(A_0)$ denote the closed subspaces consisting of the radially symmetric functions in $L^2(\Omega)$ resp. $W^{1,2}(\Omega)$, $\mathcal{D}(-\triangle + q)$. The operator A_0 is defined as the restriction of $-\triangle + q$ to $\mathcal{D}(A_0)$, and therefore again a selfadjoint semibounded operator in $\mathcal{H}$. Under condition ($\mathrm{I3}_\Omega$) we can establish the analogue of Lemma 3.1:

Lemma 4.1

(a) The map $B : X \to \mathcal{L}(X, X^*)$ defined by

$$\langle B(y)v, w\rangle := \int_\Omega f(|x|, y^2)v(x)w(x)dx$$

is completely continuous, and $B(y)$ is a compact linear operator for each $y \in X$.

(b) The functional $\varphi : X \to \mathbb{R}$ *defined by*

$$\varphi(u) := \frac{1}{2} \int_{\mathbb{R}^N} \int_0^{u^2(x)} f(|x|, s)\, ds\, dx \qquad (u \in X)$$

is continuous on X*, and it satisfies condition (CC).*

The proof is similar to that of Lemma 3.1 (Of course one gets by with easier factorizations).
Again it is now clear how to define $A(y)$, $\mu_n(y)$ and $V_n(y)$ for $y \in X$ as well as ψ and $c_n(R)$ for $R > 0$. The main result of this section now reads as follows:

Theorem 4.2 *Suppose assumptions* $(I1_\Omega)$ - $(I3_\Omega)$ *and* (M_Ω) *are satisfied. Consider arbitrary* $n \in \mathbb{N}, R > 0$ *and*

$$K_R := \{y \in S_R | y \in V_n(y)\}.$$

Then we have

i) K_R *is compact and symmetric, and* $\gamma(K_R) = n$*. In particular,* $K_R \neq \emptyset$*.*

ii) $c_n = \max_{y \in K_R} \psi(y)$*.*

iii) Every $y \in K_R \cap \psi^{-1}(c_n(R))$ *is a strong solution of (4.1)-(4.2) belonging to* $C(\Omega) \cap C^2(\Omega \setminus \{0\})$ *and having precisely* n *nodal surfaces in* $\Omega \setminus \{0\}$*.*

iv) $c_n(R) < c_{n+1}(R)$*.*

Remarks. (a) For the case (i) this theorem improves the results of [8] and [10], and now our growth conditions for f given by $(\mathrm{I3}_\Omega)$ are similar to the assumptions which are commonly used in variational approaches to this problem (see e.g. [19], [24] and the references therein). In particular, in the autonomous case ($\alpha = 0$) any subcritical growth is allowed. In the case (ii), i.e. for Ω an annulus, any polynomial growth is admitted even for $N \geq 3$, so as for instance in [3], the effect of topology is manifest. Finally, for Ω an exterior domain, we *need* polynomial growth of a certain order to guarantee the properties for K_R for *all* positive values of R. If one is interested in small solutions only, then the condition $\alpha - \beta(N-1) < -\gamma$ can, as in the previous section, be replaced by $\alpha - \beta(N-1) < 0$.
(b) As in the previous section the origin is the only point at which regularity of the solutions may fail. So in the cases (ii) and (iii) we are automatically dealing with classical solutions, while in the case (i) this can be ensured under the additional assumptions that $q(|\cdot|)$ is continuous in $x = 0$ and that $\alpha - (N-2)\beta > 0$ in $(\mathrm{I3}_\Omega)$ (cf. [8],[10]).

The *proof of Theorem 4.2* is analogous to that of Theorem 3.2(b), as soon as we have ensured that condition $(EC)_{n,D}$ is valid for every $n \in \mathbb{N}$ and any bounded subset $D \subset X$. But since in each case the equation $A(y)y = \mu y$ can again be rewritten as a singular boundary value problem of Sturm-Liouville type, $A(y)$ has only simple eigenvalues. Therefore we just have to check (2.15). In case (iii) this is done exactly as in the proof of Lemma 3.3(b), whereas in the cases (i) and (ii) we have $\mu_\infty = \infty$, since the spectrum of A_0 is discrete. Therefore we only have to check that the set $\{\mu_n(y) \mid y \in D\} \subset \mathbb{R}$ is bounded. This however is an immediate corollary of Lemma 2.2.

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