Spectral shift function of the Schrödinger operator in the large coupling constant limit, II. Positive perturbations

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Abstract

For the operator $-\Delta$ in $L^2(\mathbb{R}^d)$, $d \geq 1$, a perturbation potential $V(x) \geq 0$, and a coupling constant $\alpha > 0$, the spectral shift function $\xi(\lambda; -\Delta + \alpha V, -\Delta)$ is considered. Assuming that $V(x)$ decays as $|x|^{-l}$, $l > d$, for large x, we prove that the leading term of the asymptotics of the spectral shift function as $\alpha \to \infty$ has the form $C\alpha^{d/l}$, where the coefficient $C = C(d,l)$ is explicitly computed. The asymptotics is in agreement with the phase space volume considerations. A generalization of this result is obtained by replacing $-\Delta$ by $h(-\Delta)$ for a class of functions h.

1 Introduction

1. Spectral shift function. Let H and H_0 be self-adjoint operators in a Hilbert space. Suppose that

$$H - H_0 \in \mathfrak{S}_1, \tag{1.1}$$

where $\mathfrak{S}_1$ is the trace class. Then there exists a function $\xi \in L^1(\mathbb{R})$ such that for all $\phi \in C_0^\infty(\mathbb{R})$ the following *Lifshits-Krein trace formula* holds [15, 14]:

$$\mathrm{Tr}(\phi(H) - \phi(H_0)) = \int_{-\infty}^{\infty} \xi(\lambda)\phi'(\lambda)d\lambda, \quad \phi \in C_0^\infty(\mathbb{R}). \tag{1.2}$$

The function $\xi(\lambda) = \xi(\lambda; H, H_0)$ is called the *spectral shift function* (SSF) for the pair H_0, H. The SSF is related to the scattering matrix $S(\lambda)$ of the pair H_0, H (see, e.g., [27] for the definition of the scattering matrix) by the *Birman-Krein formula* [4]:

$$\det S(\lambda) = e^{-2\pi i \xi(\lambda)}, \tag{1.3}$$

which is valid for a.e. λ in the absolutely continuous spectrum of H_0. Due to (1.3), the SSF is often called *the scattering phase*. Formula (1.3) is sometimes interpreted as the definition of ξ. For an exposition of the SSF theory and the history of the subject, see [8] or [27, §8].

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2. Schrödinger operator. Let $H_0 = -\triangle$ in the Hilbert space $L^2(\mathbb{R}^d)$, $d \geq 1$, and let H be the Schrödinger operator, $H = H_0 + V$, where V is the operator of multiplication by a real valued potential function $V(x)$. It is well known (see [28] or [21, Theorem XI.12]) that under the assumption

$$|V(x)| \leq \frac{C}{(1+|x|)^l}, \quad l > d, \tag{1.4}$$

for sufficiently large $a > 0$ and $k > 0$ one has

$$(H+a)^{-k} - (H_0+a)^{-k} \in \mathfrak{S}_1. \tag{1.5}$$

This inclusion allows one to define the SSF $\xi(\lambda; (H+a)^{-k}, (H_0+a)^{-k})$. The SSF for the pair H_0, H is then defined by

$$\xi(\lambda; H, H_0) := -\xi((\lambda+a)^{-k}; (H+a)^{-k}, (H_0+a)^{-k}). \tag{1.6}$$

Thus defined, the SSF $\xi(\lambda; H, H_0)$ still obeys the trace formula (1.2). The definition (1.6) does not depend on the choice of k and a. See [27, §8.9] and [20] for an analysis of the definition (1.6) from the general operator theoretic point of view. For $\lambda < 0$, the SSF $\xi(\lambda; H, H_0)$ reduces to the eigenvalue counting function for the Schrödinger operator H (see, e.g., [27, §8.2]).

Various results about the high energy and semiclassical asymptotic behaviour of the SSF of the Schrödinger operator are known; see, e.g., the review paper [23] and references therein. In this paper we address the less studied problem of the behaviour of the SSF in the large coupling constant limit. Namely, let $\alpha > 0$ be a parameter (coupling constant); consider $\xi(\lambda; H_0 + \alpha V, H_0)$ as $\alpha \to \infty$. For the case of non-positive perturbations, $V \leq 0$, the following formula has been proven in [19]:

$$\begin{gathered} \xi(\lambda; H_0+\alpha V, H_0) = -\alpha^{d/2} C_{1.7}(1+o(1)), \quad \alpha \to \infty, \quad V \leq 0, \quad \text{a.e. } \lambda \in \mathbb{R}, \\ C_{1.7} = (2\pi)^{-d} \omega_d \int_{\mathbb{R}^d} |V(x)|^{d/2} dx, \quad \omega_d = \operatorname{vol}\{x \in \mathbb{R}^d \mid |x| < 1\} \end{gathered} \tag{1.7}$$

(here and elsewhere a constant which first appears in formula (i.j) is denoted by $C_{i.j}$). For $\lambda < 0$, formula (1.7) reduces to the well-known Weyl law for the eigenvalue counting function of $H_0 + \alpha V$ (see, e.g., [22, Theorem XIII.79] and references therein).

3. Main result. The purpose of this paper is to give an asymptotic formula for the SSF $\xi(\lambda; H_0 + \alpha V, H_0)$ for the case of non-negative perturbations $V \geq 0$. Here we are forced to consider the potentials V with power asymptotics at infinity. More precisely, we assume that for a function $\Psi \in C(\mathbb{S}^{d-1})$, $\Psi \geq 0$, one has

$$\sup_{\widehat{x} \in \mathbb{S}^{d-1}} |V(x) - \Psi(\widehat{x})|x|^{-l}| = o(|x|^{-l}), \quad |x| \to \infty, \tag{1.8}$$

where $l > d$ and $\widehat{x} = \frac{x}{|x|}$. In the one-dimensional case, $\mathbb{S}^0 = \{-1, 1\}$ and Ψ reduces to a pair of numbers, $\Psi(-1)$, $\Psi(1)$. Note that (1.8) implies (1.4).

Our main result is

Theorem 1.1. *Let $H_0 = -\triangle$ in $L^2(\mathbb{R}^d)$, $d \geq 1$ and let $V \geq 0$ be a bounded function which satisfies (1.8) with some $\Psi \in C(\mathbb{S}^{d-1})$, $\Psi \geq 0$, and $l > d$. Then, for any $\lambda > 0$, one has*

$$\begin{gathered} \xi(\lambda; H_0+\alpha V, H_0) = \alpha^{d/l} C_{1.9}(1+o(1)), \quad \alpha \to \infty, \quad V \geq 0, \quad \lambda > 0, \\ C_{1.9} = (2\pi)^{-d} d^{-1} \int_{|p|^2 < \lambda} (\lambda - |p|^2)^{-d/l} dp \int_{\mathbb{S}^{d-1}} \Psi^{d/l}(\widehat{x}) d\widehat{x}. \end{gathered} \tag{1.9}$$

In the one-dimensional case, $\int_{\mathbb{S}^0} \Psi^{d/l}(\widehat{x})d\widehat{x} \equiv \Psi^{d/l}(-1) + \Psi^{d/l}(1)$.

4. Remarks. 1. The trace formula (1.2) defines $\xi(\lambda)$ as an element of $L^1(\mathbb{R})$. In particular, this means that $\xi(\lambda)$ is defined as a class of functions which differ from each other on a set of zero measure. Thus, the meaning of the asymptotic relation (1.9) needs some clarification.

If the L^1 function $\xi(\lambda)$ coincides a.e. with a continuous function $\widetilde{\xi}(\lambda)$, then it is natural to identify the SSF with $\widetilde{\xi}(\lambda)$. This is exactly the situation we have in hand. If V satisfies (1.4), then $\xi(\lambda; H_0 + V, H_0)$ coincides a.e. with a continuous function (see Remark 2.2). Denote this function by $\widetilde{\xi}(\lambda; H_0 + V, H_0)$. Strictly speaking, in the l.h.s. of (1.9) we should have written $\widetilde{\xi}(\lambda; H_0 + \alpha V, H_0)$ instead of $\xi(\lambda; H_0 + \alpha V, H_0)$.

2. Formula (1.9) explains why for $V \geq 0$ we are forced to consider the more restrictive class (1.8) of potentials V (as compared to (1.4)): the order d/l of α in (1.9) is determined by the asymptotic behaviour of V. Note that the order $d/2$ of α in (1.7) does not depend on V. In a certain sense, the order of the asymptotics (1.7) is determined by the region of large momenta, whereas the order of (1.9) is determined by the region of large coordinates.

3. Following the proof, one can check that the asymptotics (1.9) is uniform in λ on every compact subset of $(0, \infty)$.

4. In the case of the potential V of a variable sign, one would expect a superposition of the terms of the type $-C_{1.7}\alpha^{d/2}$ and $C_{1.9}\alpha^{d/l}$ in the asymptotics of the SSF. If

$$\frac{d}{2} > \frac{d}{l}, \tag{1.10}$$

then it is natural to expect the leading term of the asymptotics to have the order $d/2$ and be 'generated' by the negative part of V. This is indeed the case; it has been proven in [19] that for V of a variable sign, under the assumption (1.10) the asymptotic formula (1.7) holds (in a certain average sense) with $C_{1.7}$ replaced by

$$(2\pi)^{-d}\omega_d \int_{V(x)<0} |V(x)|^{d/2} dx.$$

Note that, since $l > d$, the inequality (1.10) always holds if $d \geq 2$.

Related results about the SSF for perturbations of variable sign can be found in [24].

5. In [26], the following bound for the SSF has been obtained:

$$|\xi(\lambda; H_0 + \alpha V, H_0)| \leq C(\lambda)\alpha^{d/l}, \quad V \geq 0. \tag{1.11}$$

Theorem 1.1 shows that this bound has a sharp order in α. Note that the proof of (1.11) was based on a hypothesis (Assumption 4.3 in [26]) which has not been proven so far. In fact, the main technical result of the present paper (Theorem 4.4) provides the proof of this hypothesis.

6. The asymptotic coefficient $C_{1.9}$ can be interpreted in terms of the phase space volume. An elementary calculation shows that

$$C_{1.9} = (2\pi)^{-d} \lim_{\alpha\to\infty} \alpha^{-d/l} \operatorname{vol}\{(x,p) \in \mathbb{R}^{2d} \mid |p|^2 < \lambda < |p|^2 + \alpha V(x)\}.$$

Another compact form for $C_{1.9}$ is

$$C_{1.9} = \int_{\mathbb{R}^d} \big(\tau(\lambda) - \tau(\lambda - \Psi(\widehat{x})|x|^{-l})\big) dx, \tag{1.12}$$

where τ is the integrated density of states of $-\triangle$,

$$\tau(\lambda) = (2\pi)^{-d}\operatorname{vol}\{p \in \mathbb{R}^d \mid |p|^2 < \lambda\} = (2\pi)^{-d}\omega_d(\max\{\lambda, 0\})^{d/2}.$$

It is instructive to compare the above formulae with the discussion in [23, §1] and with the result of [2, Theorem 2.1].

5. Generalization. In Theorem 1.1, one can replace the operator $H_0 = -\triangle$ by the operator $H_0 = h(-\triangle)$, where h is a sufficiently regular function. More precisely, we assume that

$$h \in C^2(\mathbb{R}), \quad h(0) = 0, \quad h'(r) > 0 \quad \forall r > 0, \tag{1.13}$$

$$h(r) \geq Cr^\gamma, \quad C > 0, \quad \gamma > 0 \quad \text{for all large } r > 0. \tag{1.14}$$

Define $H_0 = h(-\triangle)$ according to the spectral theorem, and let $H = H_0 + V$, where V obeys (1.4). By (1.14), the relation (1.5) holds for large enough $a > 0$ and $k > 0$ (see [28]). Thus, the SSF $\xi(\lambda; H, H_0)$ is well defined by (1.6).

Theorem 1.2. *Let h satisfy* (1.13) *and* (1.14), *and let $H_0 = h(-\triangle)$ in $L^2(\mathbb{R}^d)$, $d \geq 1$. Let $V \geq 0$ be a bounded function which satisfies* (1.8) *with some $\Psi \in C(\mathbb{S}^{d-1})$, $\Psi \geq 0$, and $l > d$. Then, for a.e. $\lambda > 0$, one has*

$$\xi(\lambda; H_0 + \alpha V, H_0) = \alpha^{d/l} C_{1.15}(1 + o(1)), \quad \alpha \to \infty,$$

$$C_{1.15} = (2\pi)^{-d} d^{-1} \int_{h(|p|^2)<\lambda} (\lambda - h(|p|^2))^{-d/l} dp \int_{\mathbb{S}^{d-1}} \Psi^{d/l}(\widehat{x}) d\widehat{x}. \tag{1.15}$$

The constant $C_{1.15}$ can be written in the form (1.12) with

$$\tau(\lambda) = (2\pi)^{-d}\operatorname{vol}\{p \in \mathbb{R}^d \mid h(|p|^2) < \lambda\}.$$

In order to prove Theorem 1.2, one needs to make only minor modifications in the proof of Theorem 1.1. In Sections 2–8 we explicitly prove Theorem 1.1 and in the end of Section 7 we comment on these modifications. One can relax the assumptions (1.13), (1.14) in various directions.

6. Notation. We denote $D = -i\nabla$; in this notation, $H_0 = D^2$. We will use the notation $\chi(x^2 < a)$, $\chi(x^2 > a)$, etc. for the characteristic function of $\{x \in \mathbb{R}^d \mid |x|^2 < a\}$, $\{x \in \mathbb{R}^d \mid |x|^2 > a\}$, etc. Then, $\chi(D^2 < a)$, $\chi(D^2 > a)$, etc. are the spectral projections of D^2 corresponding to the intervals $(-\infty, a)$, (a, ∞), etc. We write $\langle x\rangle = (1 + |x|^2)^{1/2}$. For $p \in \mathbb{R}^d$, we shall write p^2 instead of $|p|^2$ for brevity. We do not distinguish between a function of $x \in \mathbb{R}^d$ and an operator of multiplication by this function in $L^2(\mathbb{R}^d)$. We assume the following convention about the signs $\pm$: if in any statement we use double indices '$\pm$', then this statement should be interpreted as a pair of statements, in one of which all the indices take the value '+', and in another — the value '−'. For a real-valued function μ, we denote $\mu_\pm = (|\mu| \pm \mu)/2$. Finally, by C (possibly with subscripts) we denote constants, whose value is of no importance for us; these constants may depend on d, l, V, λ and other auxiliary parameters, and this dependence is suppressed in the notation.

7. Let us describe the technique and the structure of the paper. Our main operator theoretic tool is a formula representation from [19] for the spectral shift function. This representation reduces the proof of Theorem 1.1 to the problem of computing the asymptotics of the spectrum of the sandwiched resolvent $\sqrt{V}(H_0 - \lambda - i0)^{-1}\sqrt{V}$. This reduction is carried out in Section 2.

In Section 3 for the ease of further reference we collect the necessary information about the weak Schatten classes of compact operators.

In the rest of the paper, Sections 4–8, we compute the asymptotics of the spectrum of the operator $\sqrt{V}(H_0-\lambda-i0)^{-1}\sqrt{V}$. The main difficulty here is the singularity of the symbol of $(H_0-\lambda-i0)^{-1}$. In Section 4 we consider the operator $\sqrt{V}(H_0-\lambda-i0)^{-1}\chi(|H_0-\lambda|>\varepsilon)\sqrt{V}$ ($\varepsilon>0$ small) with the singularity removed; for this operator the required asymptotics can be obtained by employing the result of [6]. In Sections 5–8, we show that removing the singularity as above does not affect the spectral asymptotics. In this part of the paper, we find it convenient to use the language of variational quotients (VQ); the translation of the problem into this language is given in Section 5.

2 Representation for the SSF

Let H_0 and V be as in Theorem 1.1. Here and in the rest of the paper we use the notation $W=\sqrt{V}$. For $\operatorname{Im} z>0$ introduce the compact operators

$$T(z)=W(H_0-z)^{-1}W,$$
$$A(z)=\operatorname{Re} T(z)=(T(z)+T(z)^*)/2, \quad B(z)=\operatorname{Im} T(z)=(T(z)-T(z)^*)/(2i).$$

By (1.4), *the limit $T(\lambda+i0)$ exists in the operator norm for all $\lambda>0$* (see [1] or [22, §XIII.8]; in fact, this is true under the condition $l>1$ in (1.4) instead of $l>d$). The operators $A(\lambda)\equiv A(\lambda+i0)$ and $B(\lambda)\equiv B(\lambda+i0)$ are compact and self-adjoint.

For any compact self-adjoint operator M, we use the notation

$$n_\pm(s,M)=\sharp\{n\mid \pm\lambda_n(M)>s\}, \quad s>0, \tag{2.1}$$

where $\lambda_n(M)$ are the eigenvalues of M, enumerated with the multiplicities taken into account. Similarly,

$$n(s,M)=\sharp\{n\mid |\lambda_n(M)|>s\}, \quad s>0. \tag{2.2}$$

The following representation for the SSF is valid:

$$\xi(\lambda;H_0+\alpha V,H_0)=\frac{1}{\pi}\int_{-\infty}^{\infty} n_-(\alpha^{-1},A(\lambda)+tB(\lambda))(1+t^2)^{-1}dt. \tag{2.3}$$

The representation (2.3) has been found in [18] in the framework of general operator theory. Application to the Schrödinger operator has been discussed in [19]. A similar formula is valid for negative perturbations:

$$\xi(\lambda;H_0-\alpha V,H_0)=-\frac{1}{\pi}\int_{-\infty}^{\infty} n_+(\alpha^{-1},A(\lambda)+tB(\lambda))(1+t^2)^{-1}dt.$$

A more general formula representation is available now (see [11, 20]), which accounts for perturbations V of a variable sign.

Formula (2.3) allows one to reduce the problem of computing the asymptotics of $\xi(\lambda;H_0+\alpha V,H_0)$ as $\alpha\to\infty$ to computing the asymptotics of the spectra of compact self-adjoint operators $A(\lambda)$, $B(\lambda)$. This follows from the simple lemma below, which has been proven in [19].

Lemma 2.1. *[19, Lemma 4.4] Let A and B be compact self-adjoint operators such that for some $\varkappa > \sigma > 0$, $\sigma < 1$, one has*

$$\lim_{s\to 0} s^{\varkappa} n_-(s, A) = C_{2.4}, \quad \limsup_{s\to 0} s^{\sigma} n(s, B) < \infty. \tag{2.4}$$

Then

$$\lim_{\alpha\to\infty} \alpha^{-\varkappa} \frac{1}{\pi} \int_{-\infty}^{\infty} n_-(\alpha^{-1}, A + tB)(1 + t^2)^{-1} dt = C_{2.4}.$$

The rest of the present paper is devoted to the proof of the following two relations:

$$\lim_{s\to 0} s^{d/l} n_-(s, A(\lambda)) = C_{1.9}, \tag{2.5}$$

$$\lim_{s\to 0} s^{\frac{d-1}{l-1}} n(s, B(\lambda)) < \infty, \quad d \geq 2. \tag{2.6}$$

Since $\frac{d-1}{l-1} < \frac{d}{l}$, for $d \geq 2$ formula (1.9) will follow from the representation (2.3), Lemma 2.1 and (2.5), (2.6). For $d = 1$ it is easy to see that $\operatorname{rank} B(\lambda) = 2$, and therefore

$$\limsup_{s\to 0} s^{\sigma} n_-(s, B(\lambda)) < \infty \quad \text{for any } \sigma > 0, \quad d = 1.$$

Thus, for $d = 1$ formula (1.9) will follow from the representation (2.3), Lemma 2.1 and (2.5).

Remark 2.2. It is easy to see that $A(\lambda)$ is continuous in $\lambda > 0$ in the operator norm, and $B(\lambda)$ is continuous in $\lambda > 0$ in the trace norm. By [18, Lemma 2.5], it follows that the r.h.s. of (2.3) is continuous in $\lambda > 0$ as long as -1 is not an eigenvalue of the operator $\alpha T(\lambda) = \alpha A(\lambda) + i\alpha B(\lambda)$. But under the assumption (1.4), -1 is an eigenvalue of $\alpha T(\lambda)$ only if λ is an eigenvalue of $H_0 + \alpha V$ (see [1] or [22, §XIII.8]). Since $H_0 + \alpha V$ has no positive eigenvalues (see [13] or [22, §XIII.13]), we obtain that the r.h.s. of (2.3) is continuous in $\lambda > 0$. Moreover, the same reasoning shows that the r.h.s. of (2.3) is continuous in $\alpha > 0$.

Note that results on smoothness of the SSF are well known (see, e.g. [23] and references therein), but usually they appear in the literature under much more restrictive assumptions on V than (1.4).

3 Classes $\boldsymbol{\Sigma}_{\varkappa}$ of compact operators

In this section we collect the necessary information about the weak Schatten classes $\boldsymbol{\Sigma}_{\varkappa}$ of compact operators. We use the review paper [5] as a main source of reference; however, most of the material mentioned here (apart from Proposition 3.3 and the inequalities (3.5), (3.6)) can be found, for example, in the monograph [12] (see also [25]).

1. For a compact operator A in a Hilbert space, its singular numbers are denoted by $s_n(A) = \sqrt{\lambda_n(A^*A)}$. For $\varkappa > 0$, the class $\boldsymbol{\Sigma}_{\varkappa}$ consists of all compact operators A such that

$$|A|_{\varkappa} \equiv \sup_n n^{1/\varkappa} s_n(A) < \infty. \tag{3.1}$$

The functional (3.1) defines a quasinorm in $\boldsymbol{\Sigma}_{\varkappa}$. With respect to this quasinorm, the space $\boldsymbol{\Sigma}_{\varkappa}$ is non-separable (unless the underlying Hilbert space is finite-dimensional) and complete. For $\varkappa > 1$ the space $\boldsymbol{\Sigma}_{\varkappa}$ can be equipped with a norm, which is different from, but equivalent to the quasinorm $|\cdot|_\varkappa$; we will not need the explicit formula for this norm. ¿From the definition of $s_n(A)$ it follows that

$$A^*A \in \mathbf{\Sigma}_{\varkappa/2} \quad \Leftrightarrow \quad A \in \mathbf{\Sigma}_\varkappa. \tag{3.2}$$

2. Similarly to (2.1) and (2.2), for a compact (not necessarily self-adjoint) operator A we denote

$$n(s,A) = \sharp\{n \mid s_n(A) > s\}, \quad s > 0.$$

Note that this notation is consistent with (2.2), as for a compact self-adjoint operator A one has $s_n(A) = |\lambda_n(A)|$.

For a compact operator A we consider the following functionals:

$$\begin{aligned} \Delta_\varkappa(A) &= \limsup_{s\to 0} s^\varkappa n(s,A), \\ \delta_\varkappa(A) &= \liminf_{s\to 0} s^\varkappa n(s,A). \end{aligned} \tag{3.3}$$

For $A \in \mathbf{\Sigma}_\varkappa$ one has

$$s^\varkappa n(s,A) \le |A|_\varkappa^\varkappa, \quad s > 0,$$

and so the functionals (3.3) are finite on $\mathbf{\Sigma}_\varkappa$. Conversely, if $\Delta_\varkappa(A) < \infty$, then $A \in \mathbf{\Sigma}_\varkappa$.

For a compact self-adjoint operator A, we consider also the following functionals:

$$\begin{aligned} \Delta_\varkappa^{(\pm)}(A) &= \limsup_{s\to 0} s^\varkappa n_\pm(s,A), \\ \delta_\varkappa^{(\pm)}(A) &= \liminf_{s\to 0} s^\varkappa n_\pm(s,A), \end{aligned} \tag{3.4}$$

where $n_\pm$ is defined in (2.1). The functionals (3.4) are finite on $\mathbf{\Sigma}_\varkappa$. If $\Delta_\varkappa^{(+)}(A) < \infty$ and $\Delta_\varkappa^{(-)}(A) < \infty$, then $A \in \mathbf{\Sigma}_\varkappa$. However, any one of the conditions $\Delta_\varkappa^{(+)}(A) < \infty$, $\Delta_\varkappa^{(-)}(A) < \infty$ alone does not imply that $A \in \mathbf{\Sigma}_\varkappa$; in fact, later we will deal with the case $\Delta_\varkappa^{(-)}(A) < \infty$, $A \notin \mathbf{\Sigma}_\varkappa$ — see Remark 4.6.

3. The functionals $\Delta_\varkappa^{(\pm)}$, $\delta_\varkappa^{(\pm)}$ are continuous in $\mathbf{\Sigma}_\varkappa$. This follows from the inequalities (see [5], eq. (4), (6))

$$|(\Delta_\varkappa^{(\pm)}(A_2))^{1/(\varkappa+1)} - (\Delta_\varkappa^{(\pm)}(A_1))^{1/(\varkappa+1)}| \le (\Delta_\varkappa(A_2 - A_1))^{1/(\varkappa+1)} \le |A_2 - A_1|_\varkappa^{\varkappa/(\varkappa+1)}, \tag{3.5}$$

$$|(\delta_\varkappa^{(\pm)}(A_2))^{1/(\varkappa+1)} - (\delta_\varkappa^{(\pm)}(A_1))^{1/(\varkappa+1)}| \le (\Delta_\varkappa(A_2 - A_1))^{1/(\varkappa+1)} \le |A_2 - A_1|_\varkappa^{\varkappa/(\varkappa+1)}. \tag{3.6}$$

Similarly, the functionals $\Delta_\varkappa$ and $\delta_\varkappa$ are continuous in $\mathbf{\Sigma}_\varkappa$.

If A is compact and B_1, B_2 are bounded, then $s_n(B_1AB_2) \le \|B_1\|\|B_2\| s_n(A)$. If follows that

$$A \in \mathbf{\Sigma}_\varkappa, \quad B_1,\, B_2 \text{ are bounded } \Rightarrow B_1AB_2 \in \mathbf{\Sigma}_\varkappa, \tag{3.7}$$

$$|B_1AB_2|_\varkappa \le \|B_1\|\|B_2\||A|_\varkappa. \tag{3.8}$$

Similarly, if $A = A^*$ is compact and B is bounded, then

$$\Delta_\varkappa^{(\pm)}(B^*AB) \le \|B\|^{2\varkappa} \Delta_\varkappa^{(\pm)}(A), \tag{3.9}$$

$$\delta_\varkappa^{(\pm)}(B^*AB) \le \|B\|^{2\varkappa} \delta_\varkappa^{(\pm)}(A). \tag{3.10}$$

Write $A_1 \geq A_2$ if $(A_1 f, f) \geq (A_2 f, f)$ for all vectors f. Due to the variational properties of eigenvalues, the functionals $\Delta_\varkappa^{(\pm)}$, $\delta_\varkappa^{(\pm)}$ are monotone with respect to the relation '$\geq$'. In particular, we will need the following statement:

$$A_1 \geq A_2 \quad \Rightarrow \quad \Delta_\varkappa^{(-)}(A_1) \leq \Delta_\varkappa^{(-)}(A_2). \tag{3.11}$$

4. The class $\mathbf{\Sigma}_\varkappa^0 \subset \mathbf{\Sigma}_\varkappa$ consists of all operators A such that $s_n(A) = o(n^{-1/\varkappa})$. One can prove that $\mathbf{\Sigma}_\varkappa^0$ is closed in $\mathbf{\Sigma}_\varkappa$. The space $\mathbf{\Sigma}_\varkappa^0$ is separable and coincides with the closure in $\mathbf{\Sigma}_\varkappa$ of the set of finite rank operators. It is easy to see that $A \in \mathbf{\Sigma}_\varkappa^0$ if and only if $\Delta_\varkappa(A) = 0$. One has

$$A^* A \in \mathbf{\Sigma}_{\varkappa/2}^0 \quad \Leftrightarrow \quad A \in \mathbf{\Sigma}_\varkappa^0. \tag{3.12}$$

If $A \in \mathbf{\Sigma}_\nu$, then $A \in \mathbf{\Sigma}_\varkappa^0$ for any $\varkappa > \nu$.

The A. Horn's inequality for singular numbers of a product of two operators (see [16] or [12]) yields

Proposition 3.1. *If $A_1 \in \mathbf{\Sigma}_\varkappa$ and $A_2 \in \mathbf{\Sigma}_\nu$, then $A_1 A_2 \in \mathbf{\Sigma}_\mu$, where $\frac{1}{\mu} = \frac{1}{\varkappa} + \frac{1}{\nu}$. If, moreover, $A_1 \in \mathbf{\Sigma}_\varkappa^0$ or $A_2 \in \mathbf{\Sigma}_\nu^0$, then $A_1 A_2 \in \mathbf{\Sigma}_\mu^0$.*

The following statement (which has first appeared in [6]) is a modification of the well known result (due to H. Weyl and K. Fan) about the asymptotics of the singular numbers of a sum of two operators (see, e.g., [12], Theorem 2.3).

Proposition 3.2. *Let A_1, A_2 be self-adjoint operators from the class $\mathbf{\Sigma}_\varkappa$. If $A_1 - A_2 \in \mathbf{\Sigma}_\varkappa^0$, then*

$$\Delta_\varkappa^{(\pm)}(A_1) = \Delta_\varkappa^{(\pm)}(A_2), \quad \delta_\varkappa^{(\pm)}(A_1) = \delta_\varkappa^{(\pm)}(A_2).$$

The proof can be obtained by a direct application of (3.5), (3.6).

5. Finally, we will need the following statement from [5].

Proposition 3.3. *[5, Theorem 3] Let A_1, A_2 be self-adjoint operators from the class $\mathbf{\Sigma}_\varkappa$. Suppose that $\Delta_\varkappa^{(\pm)}(A_1) = \delta_\varkappa^{(\pm)}(A_1)$, $\Delta_\varkappa^{(\pm)}(A_2) = \delta_\varkappa^{(\pm)}(A_2)$, and $A_1^* A_2 \in \mathbf{\Sigma}_{\varkappa/2}^0$, $A_1 A_2^* \in \mathbf{\Sigma}_{\varkappa/2}^0$. Then*

$$\Delta_\varkappa^{(\pm)}(A_1 + A_2) = \delta_\varkappa^{(\pm)}(A_1 + A_2) = \Delta_\varkappa^{(\pm)}(A_1) + \Delta_\varkappa^{(\pm)}(A_2).$$

4 Proof of (2.5) and (2.6)

In this section we reduce the proof of (2.5) and (2.6) to several simpler statements, which we will prove in the rest of the paper.

First, we state a result which is a 'spherically symmetric version' of (2.6).

Lemma 4.1. *Let $d \geq 2$. For any $l > d$ and $\lambda > 0$, one has*

$$\operatorname{Im}\left(\langle x\rangle^{-l/2}(D^2 - \lambda - i0)^{-1}\langle x\rangle^{-l/2}\right) \in \mathbf{\Sigma}_{\frac{d-1}{l-1}}. \tag{4.1}$$

We prove Lemma 4.1 in Sections 5, 6.

Proof of (2.6): Write $W(x) = X(x)\langle x\rangle^{-l/2}$ with X bounded. Now (2.6) follows from (3.7) and Lemma 4.1. ∎

Next, we state a result from [6] (Theorem 2 and following this theorem Remarks 1–3).

Proposition 4.2. *Let $a, b \in L^\infty(\mathbb{R}^d)$, a, b have compact support. Let $V \geq 0$ be a bounded function which satisfies* (1.8) *with some $\Psi \in C(\mathbb{S}^{d-1})$, $\Psi \geq 0$, and $l > 0$. Then*

$$a(D)Vb(D) \in \mathbf{\Sigma}_{d/l}, \tag{4.2}$$

$$\begin{aligned} \Delta_{d/l}(a(D)Vb(D)) &= \delta_{d/l}(a(D)Vb(D)) \\ &= (2\pi)^{-d}d^{-1}\int_{\mathbb{R}^d}(a(p)b(p))^{d/l}dp\int_{\mathbb{S}^{d-1}}\Psi^{d/l}(\widehat{x})d\widehat{x}. \end{aligned} \tag{4.3}$$

In fact, (4.2) and (4.3) have been proven in [6] for a much broader class of a and b, but we will not need this.

Note that, taking $a(D) = b(D)$ in (4.2) and using (3.2), we get

$$a(D)W \in \mathbf{\Sigma}_{2d/l}. \tag{4.4}$$

We will need a slightly different version of Proposition 4.2:

Proposition 4.3. *Let $\mu \in L^1(\mathbb{R}^d)$, μ have a compact support. Let V be as in Theorem 1.1 and $W = \sqrt{V}$. Then $W\mu(D)W \in \mathbf{\Sigma}_{d/l}$ and*

$$\Delta^{(\pm)}_{d/l}(W\mu(D)W) = \delta^{(\pm)}_{d/l}(W\mu(D)W) = (2\pi)^{-d}d^{-1}\int_{\mathbb{R}^d}(\mu_\pm(p))^{d/l}dp\int_{\mathbb{S}^{d-1}}\Psi^{d/l}(\widehat{x})d\widehat{x}. \tag{4.5}$$

Proof. 1. First assume that $\mu \geq 0$. Then

$$\lambda_n(W\mu(D)W) = \lambda_n(\mu^{1/2}(D)V\mu^{1/2}(D)),$$

and the result follows from Proposition 4.2.

2. For μ of a variable sign, write

$$W\mu(D)W = W\mu_+(D)W - W\mu_-(D)W.$$

By the previous step and Proposition 3.3, it suffices to check that

$$(W\mu_+(D)W)(W\mu_-(D)W) \in \mathbf{\Sigma}^0_{d/(2l)}. \tag{4.6}$$

3. Write the operator in (4.6) as

$$(W\mu_+^{1/2}(D))(\mu_+^{1/2}(D)V\mu_-^{1/2}(D))(\mu_-^{1/2}(D)W).$$

By (4.4), the first and the third term in the above product belong to $\mathbf{\Sigma}_{2d/l}$. By (4.3), the second term belongs to $\mathbf{\Sigma}^0_{d/l}$ (as $\mu_+\mu_- \equiv 0$). Thus, applying Proposition 3.1 to the above product, we get (4.6). ∎

Next, we state the main technical result of the paper.

Theorem 4.4. *For any $l > d$ and $\lambda > 0$, one has*

$$\operatorname{Re}(\langle x\rangle^{-l/2}(D^2-\lambda-i0)^{-1}\chi(D^2<2\lambda)\langle x\rangle^{-l/2}) \in \mathbf{\Sigma}_{d/l}, \tag{4.7}$$

$$\Delta_{d/l}\big(\operatorname{Re}(\langle x\rangle^{-l/2}(D^2-\lambda-i0)^{-1}\chi(|D^2-\lambda|<\varepsilon)\langle x\rangle^{-l/2})\big) \to 0 \text{ as } \varepsilon \to 0. \tag{4.8}$$

We prove this theorem in Sections 5, 7–8. This theorem will allow us to get rid of the singularity of the symbol of $A(\lambda)$. Note that (4.7) has been conjectured in [26, Assumption 4.3].

Finally, we will need the following technical result.

Proposition 4.5. *Let V be as in Theorem 1.1 and $W = \sqrt{V}$. Then, for any $\lambda > 0$, one has*

$$\chi(D^2 < \lambda) W \chi(D^2 > \lambda) \in \mathbf{\Sigma}^0_{2d/l}. \tag{4.9}$$

We postpone the proof of this proposition till the end of this section.

Proof of (2.5): 1. We will prove separately the following two inequalities:

$$\Delta^{(-)}_{d/l}(A(\lambda)) \leq C_{1.9}, \tag{4.10}$$

$$\delta^{(-)}_{d/l}(A(\lambda)) \geq C_{1.9}. \tag{4.11}$$

Clearly, (4.10) and (4.11) imply (2.5).

2. In order to prove (4.10), we first notice that

$$A(\lambda) \geq \operatorname{Re}(W(D^2 - \lambda - i0)^{-1}\chi(D^2 < 2\lambda)W).$$

By (3.11), this yields

$$\Delta^{(-)}_{d/l}(A(\lambda)) \leq \Delta^{(-)}_{d/l}\big(\operatorname{Re}(W(D^2 - \lambda - i0)^{-1}\chi(D^2 < 2\lambda)W)\big).$$

Next, we factorize $W(x) = \langle x\rangle^{-l/2} X(x)$ with bounded X and for any $\varepsilon > 0$ write

$$\begin{aligned}\operatorname{Re}(W(D^2 - \lambda - i0)^{-1}\chi(D^2 < 2\lambda)W) &= W(D^2 - \lambda)^{-1}\chi(D^2 < 2\lambda)\chi(|D^2 - \lambda| > \varepsilon)W \\ &+ X \operatorname{Re}(\langle x\rangle^{-l/2}(D^2 - \lambda - i0)^{-1}\chi(|D^2 - \lambda| < \varepsilon)\langle x\rangle^{-l/2})X.\end{aligned}$$

By (4.8) and (3.9),

$$\Delta_{d/l}(X \operatorname{Re}(\langle x\rangle^{-l/2}(D^2 - \lambda - i0)^{-1}\chi(|D^2 - \lambda| < \varepsilon)\langle x\rangle^{-l/2})X) \to 0 \quad \text{as } \varepsilon \to 0.$$

By (3.5), it follows

$$\Delta^{(-)}_{d/l}(A(\lambda)) = \Delta^{(-)}_{d/l}(W(D^2 - \lambda)^{-1}\chi(D^2 < 2\lambda)\chi(|D^2 - \lambda| > \varepsilon)W) + o(1), \quad \varepsilon \to 0.$$

Finally, by Proposition 4.3,

$$\begin{aligned}&\Delta^{(-)}_{d/l}(W(D^2 - \lambda)^{-1}\chi(D^2 < 2\lambda)\chi(|D^2 - \lambda| > \varepsilon)W) \\ &\qquad = (2\pi)^{-d} d^{-1} \int_{p^2 < \lambda - \varepsilon} (\lambda - p^2)^{-d/l} dp \int_{\mathbb{S}^{d-1}} \Psi^{d/l}(\widehat{x}) d\widehat{x}.\end{aligned}$$

Letting $\varepsilon \to 0$, we get (4.10).

3. Let us prove (4.11). Denote for brevity $E_1 = \chi(D^2 < 2\lambda)$, $E_2 = \chi(D^2 > 2\lambda)$. By (3.10),

$$\delta^{(-)}_{d/l}(A(\lambda)) \geq \delta^{(-)}_{d/l}(E_1 A(\lambda) E_1). \tag{4.12}$$

For any $\varepsilon > 0$, write

$$E_1 A(\lambda) E_1 = E_1 W (D^2 - \lambda)^{-1} \chi(|D^2 - \lambda| > \varepsilon) W E_1 \\ + E_1 \mathrm{Re}\,(W(D^2 - \lambda - i0)^{-1} \chi(|D^2 - \lambda| < \varepsilon) W) E_1. \quad (4.13)$$

By (4.4) with $a(D) = \chi(D^2 < 2\lambda)$, one has $E_1 W \in \mathbf{\Sigma}_{2d/l}$. Thus, by Proposition 3.1, the first term in the r.h.s. of (4.13) belongs to $\mathbf{\Sigma}_{d/l}$. The second term belongs to $\mathbf{\Sigma}_{d/l}$ by Theorem 4.4. Thus, $E_1 A(\lambda) E_1 \in \mathbf{\Sigma}_{d/l}$. Next, as on the previous step of the proof, using (4.8), (3.6) and (3.9), we get

$$\delta^{(-)}_{d/l}(E_1 A(\lambda) E_1) = \delta^{(-)}_{d/l}(E_1 W (D^2 - \lambda)^{-1} \chi(|D^2 - \lambda| > \varepsilon) W E_1) + o(1), \quad \varepsilon \to 0. \quad (4.14)$$

4. Denote $\mu(D) = (D^2 - \lambda)^{-1} \chi(|D^2 - \lambda| > \varepsilon)$. Let us check that

$$E_1 W \mu(D) W E_1 - W E_1 \mu(D) E_1 W \in \mathbf{\Sigma}^0_{d/l}. \quad (4.15)$$

One has

$$E_1 W \mu(D) W E_1 - W E_1 \mu(D) E_1 W = (E_1 W E_2) \mu(D) (E_2 W E_1) - (E_2 W E_1) \mu(D) (E_1 W E_1) \\ - (E_1 W E_1) \mu(D) (E_1 W E_2) - (E_2 W E_1) \mu(D) (E_1 W E_2).$$

By Proposition 4.5 and Proposition 3.1, all the terms in the r.h.s. of the last equation belong to $\mathbf{\Sigma}^0_{d/l}$.

5. By (4.15), Proposition 3.2 and Proposition 4.3,

$$\delta^{(-)}_{d/l}(E_1 W (D^2 - \lambda)^{-1} \chi(|D^2 - \lambda| > \varepsilon) W E_1) = \delta^{(-)}_{d/l}(W E_1 (D^2 - \lambda)^{-1} \chi(|D^2 - \lambda| > \varepsilon) E_1 W) \\ = (2\pi)^{-d} d^{-1} \int_{p^2 < \lambda - \varepsilon} (\lambda - p^2)^{-d/l} dp \int_{\mathbb{S}^{d-1}} \Psi^{d/l}(\widehat{x}) d\widehat{x}.$$

This, together with (4.12) and (4.14), gives (4.11). ∎

Remark 4.6. Note that (see [19]) $A(\lambda) \in \mathbf{\Sigma}_{d/2}$ and $\Delta^{(+)}_{d/2}(A(\lambda)) \neq 0$. Thus, $A(\lambda)$ does *not* belong to $\mathbf{\Sigma}_{d/l}$, unless $d/l \geq d/2$ (this can happen only for $d = 1$).

Proof of Proposition 4.5: 1. Fix a function ζ,

$$\zeta \in C^\infty(\mathbb{R}^d), \quad \zeta(x) \geq 0 \text{ for all } x \in \mathbb{R}^d, \quad \zeta(x) = 1 \text{ for } |x| \geq 2, \quad \zeta(x) = 0 \text{ for } |x| \leq 1. \quad (4.16)$$

Denote $W_0(x) = \Psi^{1/2}(\widehat{x}) |x|^{-l/2} \zeta(x)$. By (1.8), for any $\varepsilon > 0$ one can write $W = W_0 + W_\varepsilon + \widetilde{W}_\varepsilon$, where $\widetilde{W}_\varepsilon$ has a compact support and

$$|W_\varepsilon(x)| \leq \varepsilon |x|^{-l/2} \zeta(x). \quad (4.17)$$

By Proposition 4.2, $\chi(D^2 < \lambda) \widetilde{W}_\varepsilon^2 \chi(D^2 < \lambda) \in \mathbf{\Sigma}^0_{d/l}$ and so, by (3.12), one has $\chi(D^2 < \lambda) \widetilde{W}_\varepsilon \in \mathbf{\Sigma}^0_{2d/l}$. By (4.4), $\chi(D^2 < \lambda) |x|^{-l/2} \zeta(x) \in \mathbf{\Sigma}_{2d/l}$; therefore, by (3.8) and (4.17),

$$|\chi(D^2 < \lambda) W_\varepsilon|_{2d/l} \to 0 \text{ as } \varepsilon \to 0.$$

Thus, since $\mathbf{\Sigma}^0_{2d/l}$ is closed in $\mathbf{\Sigma}_{2d/l}$, it is sufficient to prove that

$$\chi(D^2 < \lambda) W_0 \chi(D^2 > \lambda) \in \mathbf{\Sigma}^0_{2d/l}. \tag{4.18}$$

2. By (3.8),

$$|\chi(D^2 < \lambda) W_0 \chi(D^2 > \lambda)|_{2d/l} \le |\chi(D^2 < \lambda)|x|^{-l/2}\zeta(x)|_{2d/l} \|\Psi^{1/2}\|_{L^\infty(\mathbb{S}^{d-1})}.$$

Thus, approximating $\Psi^{1/2}$ in $L^\infty(\mathbb{S}^{d-1})$ by smooth functions, we approximate the operator (4.18) in $\mathbf{\Sigma}_{2d/l}$. Since $\mathbf{\Sigma}^0_{2d/l}$ is closed in $\mathbf{\Sigma}_{2d/l}$, it is sufficient to prove (4.18) for smooth $\Psi^{1/2}$ (in fact, we will need $\Psi^{1/2} \in C^1(\mathbb{S}^{d-1})$).

3. By Proposition 4.2,

$$\chi(D^2 < \lambda) W_0 \chi(\lambda < D^2 < \lambda + R) \in \mathbf{\Sigma}^0_{2d/l},$$

as $\chi(p^2 < \lambda)\chi(\lambda < p^2 < \lambda + R) \equiv 0$. Thus, it suffices to prove that

$$\chi(D^2 < \lambda) W_0 \chi(D^2 > \lambda + R) \in \mathbf{\Sigma}^0_{2d/l} \tag{4.19}$$

for all large enough $R > 0$.

4. The operator in (4.19) is unitary equivalent to the operator

$$\chi(|x| < \sqrt{\lambda}) W_0(D) \chi(|x| > \sqrt{\lambda + R}). \tag{4.20}$$

Take $R = 4 + 4\sqrt{\lambda}$, so that $\sqrt{\lambda + R} = \sqrt{\lambda} + 2$. Then the integral kernel $K(x, y)$ of the operator (4.20) vanishes for $|x - y| \le 2$. Thus, multiplying the kernel $K(x, y)$ by $\zeta(x - y)$ (where ζ is as in (4.16)) does not change the operator (4.20).

We have arrived at the following statement. Let $\eta(x) = 1 - \zeta(x)$, let $\widehat{\eta}$ be the (unitary) Fourier transform of η, and let $W_0 * \widehat{\eta}$ be the convolution of W_0 and $\widehat{\eta}$. Then

$$\begin{aligned}\chi(|x| < \sqrt{\lambda}) & W_0(D) \chi(|x| > \sqrt{\lambda} + 2) \\ &= \chi(|x| < \sqrt{\lambda})(W_0(D) - (2\pi)^{-d/2}(W_0 * \widehat{\eta})(D))\chi(|x| > \sqrt{\lambda} + 2)\end{aligned} \tag{4.21}$$

Below we prove that

$$\lim_{|x| \to \infty} |x|^{l/2} |W_0(x) - (2\pi)^{-d/2}(W_0 * \widehat{\eta})(x)| = 0. \tag{4.22}$$

By Proposition 4.2, this implies

$$\chi(|x| < \sqrt{\lambda})(W_0(D) - (2\pi)^{-d/2}(W_0 * \widehat{\eta})(D)) \in \mathbf{\Sigma}^0_{2d/l}.$$

By (4.21), this yields (4.19).

5. To complete the proof, it remains to check (4.22). This is a straightforward computation. First note that $\widehat{\eta}$ is in the Schwartz class. Next, $\eta(0) = 1$ and so

$$(2\pi)^{-d/2} \int_{\mathbb{R}^d} \widehat{\eta}(y) dy = 1.$$

Thus,

$$
\begin{aligned}
|x|^{l/2}(W_0(x)-(2\pi)^{-d/2}(W_0*\widehat{\eta})(x)) &= (2\pi)^{-d/2}|x|^{l/2}\int_{\mathbb{R}^d}\widehat{\eta}(y)(W_0(x)-W_0(x-y))dy\\
&= (2\pi)^{-d/2}\int_{\mathbb{R}^d}\widehat{\eta}(y)(\Psi^{1/2}(\widehat{x})\zeta(x)-\Psi^{1/2}(\widehat{x-y})|x|^{l/2}|x-y|^{-l/2}\zeta(x-y))dy.
\end{aligned}
$$

Let us split the last integral into the sum of two integrals over $|y|\leq\sqrt{|x|}$ and $|y|\geq\sqrt{|x|}$ and check that both integrals tend to zero as $|x|\to\infty$. For the first integral, one has

$$
\begin{aligned}
\left|\int_{|y|\leq\sqrt{|x|}}(\dots)dy\right| &\leq \|\widehat{\eta}\|_{L^1}\|\Psi^{1/2}\|_{L^\infty}\sup_{|y|\leq\sqrt{|x|}}\left||x|^{l/2}|x-y|^{-l/2}\zeta(x-y)-\zeta(x)\right|\\
&\quad+\|\widehat{\eta}\|_{L^1}\sup_{|y|\leq\sqrt{|x|}}|\Psi^{1/2}(\widehat{x-y})-\Psi^{1/2}(\widehat{x})| = O(1/\sqrt{|x|}),\quad |x|\to+\infty.
\end{aligned}
$$

In the last estimate we have used the fact that $\Psi^{1/2}\in C^1(\mathbb{S}^{d-1})$.

Next,

$$
\left|\int_{|y|\geq\sqrt{|x|}}(\dots)dy\right| \leq \left(\int_{|y|\geq\sqrt{|x|}}|\widehat{\eta}(y)|dy\right)\sup_{|y|\geq\sqrt{|x|}}\left|\Psi^{1/2}(\widehat{x-y})|x|^{l/2}|x-y|^{-l/2}\zeta(x-y)-\Psi^{1/2}(\widehat{x})\zeta(x)\right|;
$$

here the first term decays faster than any power of $|x|$, and the second term is $O(|x|^{l/2})$. ∎

5 Variational Quotients

In the rest of the paper, we prove Lemma 4.1 and Theorem 4.4. In this section, we translate Lemma 4.1 and Theorem 4.4 into the language of variational quotients (VQ).

Let $\mu\in L^1(\mathbb{R}^d)$ be a real-valued compactly supported function. Consider the VQ

$$
t[u]=\frac{\int_{\mathbb{R}^d}\mu(p)|u(p)|^2dp}{\|u\|^2_{H^{l/2}}},\quad u\in\mathrm{Dom}(t)=H^{l/2}(\mathbb{R}^d). \tag{5.1}
$$

Here the norm in the Sobolev space $H^{l/2}(\mathbb{R}^d)$ is defined by

$$
\|u\|^2_{H^{l/2}}=\int_{\mathbb{R}^d}\langle p\rangle^l|\widehat{u}(p)|^2dp, \tag{5.2}
$$

where $\widehat{u}$ is the Fourier transform of u.

Consider the numerator of (5.1) as a quadratic form in the Hilbert space $\mathrm{Dom}(t)$ with the metric given by the denominator of (5.1); denote this quadratic form by $\mathfrak{t}[u,u]$. The form $\mathfrak{t}[u,u]$ generates a self-adjoint operator T (in the same Hilbert space):

$$
\mathfrak{t}[u,u]=(Tu,u)_{H^{l/2}},\quad u\in\mathrm{Dom}(T),
$$

where $(\cdot,\cdot)_{H^{l/2}}$ is the inner product, corresponding to the norm (5.2):

$$
(u,v)_{H^{l/2}}=\int_{\mathbb{R}^d}\langle p\rangle^l\widehat{u}(p)\overline{\widehat{v}(p)}dp. \tag{5.3}
$$

The *spectrum* of the VQ (5.1) is the spectrum of the operator T. Below we use the notation of the type

$$n(s,(5.1)) := n(s,T), \quad s>0,$$

where the r.h.s. is defined by (2.2). By the variational characterization of eigenvalues, one has

$$n(s,(5.1)) = \sup\{\dim \mathcal{L} \mid \mathcal{L} \subset \mathrm{Dom}(t), |t[u]| > s \quad \forall u \in \mathcal{L}\};$$

here $\mathcal{L}$ is a linear subspace with the stated properties.

It is easy to see that the operator T, generated by (5.1), is unitary equivalent to the operator $\langle x\rangle^{-l/2}\mu(D)\langle x\rangle^{-l/2}$ in the Hilbert space $L^2(\mathbb{R}^d)$. Thus,

$$n(s,\langle x\rangle^{-l/2}\mu(D)\langle x\rangle^{-l/2}) = n(s,(5.1)), \quad s>0.$$

Fix $\lambda>0$ and $\varepsilon>0$. Taking $\mu(p) = \chi(|p^2-\lambda|<\varepsilon)\mathrm{Re}\,(p^2-\lambda-i\delta)^{-1}$ and letting $\delta\to+0$, we see that the VQ

$$\frac{v.p. \int_{|p^2-\lambda|<\varepsilon}(p^2-\lambda)^{-1}|u(p)|^2dp}{\|u\|^2_{H^{l/2}}}, \quad u\in H^{l/2}(\mathbb{R}^d) \tag{5.4}$$

determines the spectrum of the operator from (4.8):

$$n\big(s, \mathrm{Re}\,(\langle x\rangle^{-l/2}(D^2-\lambda-i0)^{-1}\chi(|D^2-\lambda|<\varepsilon)\langle x\rangle^{-l/2})\big) = n(s,(5.4)). \tag{5.5}$$

In the same way, the VQ

$$\frac{(\pi/2)\lambda^{(d/2)-1}\int_{\mathbb{S}^{d-1}}|u(\sqrt{\lambda}\omega)|^2d\omega}{\|u\|^2_{H^{l/2}}}, \quad u\in H^{l/2}(\mathbb{R}^d) \tag{5.6}$$

determines the spectrum of the operator from (4.1):

$$n\big(s, \mathrm{Im}\,(\langle x\rangle^{-l/2}(D^2-\lambda-i0)^{-1}\langle x\rangle^{-l/2})\big) = n(s,(5.6)). \tag{5.7}$$

In Section 6, we prove that

$$\limsup_{s\to 0} s^{\frac{d-1}{l-1}} n(s,(5.6)) < \infty. \tag{5.8}$$

By (5.7), this yields Lemma 4.1.

In Sections 7–8 we prove that

$$\limsup_{s\to 0} s^{d/l} n(s,(5.4)) \le C_{5.9} = (2\pi)^{-d}\omega_d \int_{|p^2-\lambda|<\varepsilon} |p^2-\lambda|^{-d/l}dp. \tag{5.9}$$

By (5.5), this yields Theorem 4.4.

In dealing with variational quotients, we will use the following simple facts:

(i) Increasing the VQ leads to the increase of the counting function n:

$$t_1[u] \le t_2[u] \quad \forall u \in \mathrm{Dom}(t_1) = \mathrm{Dom}(t_2) \quad \Rightarrow \quad n(s,T_1) \le n(s,T_2) \quad \forall s>0.$$

(ii) Extending the domain of the VQ leads to the increase of the counting function n:

$$t_1[u] = t_2[u] \quad \forall u \in \mathrm{Dom}(t_1) \subset \mathrm{Dom}(t_2) \quad \Rightarrow \quad n(s,T_1) \le n(s,T_2) \quad \forall s>0.$$

6 Proof of (5.8)

Throughout this section, $d \geq 2$. The key idea of the proof of (5.8) is to reduce the VQ (5.6) to a VQ on a sphere in $\mathbb{R}^d$.

1. First, we need notation for some sets in $\mathbb{R}^d$; fix $\lambda > 0$ and denote

$$\begin{aligned} S_\lambda = \{x \in \mathbb{R}^d \mid |x| = \sqrt{\lambda}\}, \quad S_\lambda^c = \{x \in \mathbb{R}^d \mid |x| \neq \sqrt{\lambda}\}, \\ S_\lambda^i = \{x \in \mathbb{R}^d \mid |x| < \sqrt{\lambda}\}, \quad S_\lambda^e = \{x \in \mathbb{R}^d \mid |x| > \sqrt{\lambda}\} \end{aligned} \tag{6.1}$$

(the superscripts 'c', 'i', 'e' stand for 'complement', 'interior', 'exterior'). In what follows, for $d \geq 2$ we use the Sobolev classes $H^p(S_\lambda)$, $p \in \mathbb{R}$. The exact choice of the norm in $H^p(S_\lambda)$ will be of no importance for us; in all the construction, this norm can be replaced by any equivalent norm. One of the possible choices of the norm is

$$\|u\|_{H^p(S_\lambda)} = \|(I - \triangle_B)^{p/2} u\|_{L^2(S_\lambda)}, \tag{6.2}$$

where $\triangle_B$ is the Laplace-Beltrami operator on S_λ.

Proposition 6.1. *Fix $q \in \mathbb{R}$ and $p > 0$, $p > q$. Then, for the VQ*

$$\frac{\|u\|^2_{H^q(S_\lambda)}}{\|u\|^2_{H^p(S_\lambda)}}, \quad u \in H^p(S_\lambda), \tag{6.3}$$

one has

$$\limsup_{s \to 0} s^{\varkappa} n(s, (6.3)) < \infty, \quad \varkappa = \frac{d-1}{2(p-q)}. \tag{6.4}$$

Proof: For the choice (6.2) of the metric, the spectrum of the VQ (6.3) coincides with the spectrum of $(I - \triangle_B)^{q-p}$. Thus,

$$n(s, (6.3)) = n(s^{1/(p-q)}, (I - \triangle_B)^{-1}),$$

and (6.4) follows from the well-known asymptotics

$$\lim_{s \to 0} s^{(d-1)/2} n(s, (I - \triangle_B)^{-1}) = C. \quad \blacksquare$$

In this section, we will only use Proposition 6.1 for $p = (l-1)/2$, $q = 0$, but we will need the general version in Section 8.

2. Denote by γ the following restriction operator:

$$\gamma : H^{l/2}(\mathbb{R}^d) \to H^{(l-1)/2}(S_\lambda), \quad u \mapsto u\,|_{S_\lambda}.$$

As it is well known, the operator γ is bounded:

$$\|\gamma u\|_{H^{(l-1)/2}} \leq C_{6.5} \|u\|_{H^{l/2}}. \tag{6.5}$$

This enables us to estimate the VQ (5.6) as follows:

$$\frac{\|\gamma u\|^2_{L^2(\mathbb{S}^{d-1})}}{\|u\|^2_{H^{l/2}(\mathbb{R}^d)}} \leq C_{6.5}^{-2} \frac{\|\gamma u\|^2_{L^2(\mathbb{S}^{d-1})}}{\|\gamma u\|^2_{H^{(l-1)/2}(\mathbb{S}^{d-1})}}, \quad u \in H^{l/2}(\mathbb{R}^d).$$

Thus, the spectrum of (6.3) with $p = (l-1)/2$, $q = 0$ gives the estimate for the spectrum of (5.6) and so (5.8) follows from Proposition 6.1.

7 Proof of (5.9)

1. As above, we use the notation (6.1) (for $d \geq 1$) and consider $H^{l/2}(\mathbb{R}^d)$ as a Hilbert space with respect to the inner product (5.3). Let $H_0^{l/2}(S_\lambda^c)$ be the closure of $C_0^\infty(S_\lambda^c)$ in $H^{l/2}(\mathbb{R}^d)$ and let $H_\perp^{l/2}(\mathbb{R}^d)$ be the orthogonal complement of $H_0^{l/2}(S_\lambda^c)$:

$$H^{l/2}(\mathbb{R}^d) = H_0^{l/2}(S_\lambda^c) \oplus H_\perp^{l/2}(\mathbb{R}^d). \tag{7.1}$$

According to (7.1), we shall write

$$u = u_0 + u_\perp, \quad u_0 \in H_0^{l/2}(S_\lambda^c), \quad u_\perp \in H_\perp^{l/2}(\mathbb{R}^d).$$

2. We will need the following two statements.

Proposition 7.1. *Let $H_\perp^{l/2}(\mathbb{R}^d)$ be as defined above, and let $\delta \geq 0$, $\delta < (l-1)/2$. Then, for the VQ*

$$\frac{\|u_\perp\|^2_{H^{\frac{1}{2}+\delta}(\mathbb{R}^d)}}{\|u_\perp\|^2_{H^{l/2}(\mathbb{R}^d)}}, \quad u_\perp \in H_\perp^{l/2}(\mathbb{R}^d), \tag{7.2}$$

one has

$$\lim_{s \to 0} s^{\varkappa} n(s, (7.2)) = 0, \quad \forall \varkappa > \frac{d-1}{l-2\delta-1}. \tag{7.3}$$

We prove Proposition 7.1 in Section 8.

Proposition 7.2. *[10] Let, as above, $l > d$ and let μ be a real valued compactly supported function on $\mathbb{R}^d$ such that*

$$|\mu(p)| \leq C ||p| - \sqrt{\lambda}|^{-\rho}, \quad \rho < l/d. \tag{7.4}$$

Then for any $u_0 \in H_0^{l/2}(S_\lambda^c)$, the integral $\int_{\mathbb{R}^d} \mu(p)|u_0(p)|^2 dp$ converges and for the VQ

$$\frac{\int_{\mathbb{R}^d} \mu(p)|u_0(p)|^2 dp}{\|u_0\|^2_{H^{l/2}(\mathbb{R}^d)}}, \quad u_0 \in H_0^{l/2}(S_\lambda^c), \tag{7.5}$$

one has

$$\lim_{s \to 0} s^{d/l} n(s, (7.5)) = (2\pi)^{-d} \omega_d \int_{\mathbb{R}^d} |\mu(p)|^{d/l} dp. \tag{7.6}$$

For $\mu \in L^1(\mathbb{R}^d)$, Proposition 7.2 is well known (see, e.g, [7]). In this case (7.6) is also true (and is equivalent to Proposition 4.3 with $V(x) = \langle x \rangle^{-l}$) if $H_0^{l/2}(S_\lambda^c)$ is replaced by $H^{l/2}(\mathbb{R}^d)$ in (7.5).

In the form, stated above, Proposition 7.2 allows for a non-integrable singularity of μ. This situation has been studied in [3] for $d = 1$ and in [10] for $d \geq 2$. Here the space $H_0^{l/2}(S_\lambda^c)$ *cannot* be replaced by $H^{l/2}(\mathbb{R}^d)$. The construction of [3, 10] is based on the generalized Hardy inequality, which ensures that the functions $u_0(p)$ of $H_0^{l/2}(S_\lambda)$ tend to zero as p approaches S_λ.

Finally, note that in [10], the VQ of the type (7.5) was studied on a cube in $\mathbb{R}^d$, rather than on a sphere S_λ. However, Proposition 7.2 in the above stated form follows by application of the standard technique of the local change of coordinates.

3. We start proving (5.9). Let us estimate the numerator of (5.4). Fix $\tau > 0$ and $\theta_0 > 0$, $\theta_1 > 0$ so that $\theta_0 + \theta_1 = 1$, $\theta_0 < l/(2d)$, $\theta_1 < 1/2$. Then, using the decomposition (7.1) on the first step and the Cauchy inequality on the second step, we get

$$
\begin{aligned}
v.p. \int\limits_{|p^2-\lambda|<\varepsilon} (p^2-\lambda)^{-1}|u(p)|^2 dp = \int\limits_{|p^2-\lambda|<\varepsilon} (p^2-\lambda)^{-1}|u_0(p)|^2 dp + v.p. \int\limits_{|p^2-\lambda|<\varepsilon} (p^2-\lambda)^{-1}|u_\perp(p)|^2 dp \\
+2\mathrm{Re} \int\limits_{|p^2-\lambda|<\varepsilon} (p^2-\lambda)^{-1} u_\perp(p)\overline{u_0(p)} \leq \int\limits_{|p^2-\lambda|<\varepsilon} |p^2-\lambda|^{-1}|u_0(p)|^2 dp + v.p. \int\limits_{|p^2-\lambda|<\varepsilon} (p^2-\lambda)^{-1}|u_\perp(p)|^2 dp \\
+\tau \int\limits_{|p^2-\lambda|<\varepsilon} |p^2-\lambda|^{-2\theta_0}|u_0(p)|^2 dp + \frac{1}{\tau} \int\limits_{|p^2-\lambda|<\varepsilon} |p^2-\lambda|^{-2\theta_1}|u_\perp(p)|^2 dp. \quad (7.7)
\end{aligned}
$$

As it will follow from further reasoning, all the integrals in the r.h.s., apart from the second one, converge absolutely.

Below we consider separately the four VQ, corresponding to the four terms in the r.h.s. of (7.7).

4. Consider the VQ

$$
\frac{\int_{|p^2-\lambda|<\varepsilon} |p^2-\lambda|^{-1}|u_0(p)|^2 dp}{\|u\|^2_{H^{l/2}}}, \quad u \in H^{l/2}(\mathbb{R}^d). \tag{7.8}
$$

First note that, according to (7.1), the compact self-adjoint operator in $H^{l/2}(\mathbb{R}^d)$, generated by (7.8), splits into the orthogonal sum of the operator, generated by

$$
\frac{\int_{|p^2-\lambda|<\varepsilon} |p^2-\lambda|^{-1}|u_0(p)|^2 dp}{\|u_0\|^2_{H^{l/2}}}, \quad u_0 \in H_0^{l/2}(S_\lambda^c) \tag{7.9}
$$

in $H_0^{l/2}(S_\lambda^c)$ and zero operator in $H_\perp^{l/2}(\mathbb{R}^d)$. Thus,

$$
n(s,(7.8)) = n(s,(7.9)).
$$

The weight function $\mu(p) = |p^2-\lambda|^{-1}\chi(|p^2-\lambda| < \varepsilon)$ in the numerator of (7.9) satisfies (7.4) and thus, by Proposition 7.2,

$$
\lim_{s\to 0} s^{d/l} n(s,(7.9)) = C_{5.9}. \tag{7.10}
$$

5. Consider the VQ, generated by the third term in (7.7). By the same reasoning as above, it reduces to the VQ in $H_0^{l/2}(S_\lambda^c)$:

$$
\frac{\tau \int_{|p^2-\lambda|<\varepsilon} |p^2-\lambda|^{-2\theta_0}|u_0(p)|^2 dp}{\|u_0\|^2_{H^{l/2}}}, \quad u_0 \in H_0^{l/2}(S_\lambda^c). \tag{7.11}
$$

Again, by Proposition 7.2,

$$\lim_{s\to 0} s^{d/l} n(s,(7.11)) = \tau^{d/l}(2\pi)^{-d}\omega_d \int_{|p^2-\lambda|<\varepsilon} |p^2-\lambda|^{-2\theta_0 d/l}dp. \tag{7.12}$$

6. Consider the VQ, generated by the fourth term in (7.7):

$$\frac{\tau^{-1}\int_{|p^2-\lambda|<\varepsilon}|p^2-\lambda|^{-2\theta_1}|u_\perp(p)|^2dp}{\|u\|^2_{H^{l/2}}}, \quad u\in H^{l/2}(\mathbb{R}^d). \tag{7.13}$$

First note that, by the same reasoning as above, the VQ (7.13) reduces to a VQ in $H^{l/2}_\perp(\mathbb{R}^d)$. Next, let us estimate the numerator. For $r>0$, denote

$$\widetilde{u}_\perp(r) = r^{d-1}\int_{\mathbb{S}^{d-1}}|u_\perp(r\omega)|^2 d\omega. \tag{7.14}$$

Then, separating the variables in the integral, we get

$$\frac{1}{\tau}\int_{|p^2-\lambda|<\varepsilon}|p^2-\lambda|^{-2\theta_1}|u_\perp(p)|^2dp = \frac{1}{\tau}\int_{\sqrt{\lambda-\varepsilon}}^{\sqrt{\lambda+\varepsilon}}|r^2-\lambda|^{-2\theta_1}\widetilde{u}_\perp(r)dr \le \frac{1}{\tau}\|\widetilde{u}_\perp\|_{L^\infty}\int_{\sqrt{\lambda-\varepsilon}}^{\sqrt{\lambda+\varepsilon}}|r^2-\lambda|^{-2\theta_1}dr,$$

where the norm of $\widetilde{u}_\perp$ is taken in $L^\infty(\sqrt{\lambda-\varepsilon},\sqrt{\lambda+\varepsilon})$. By the embedding theorem (see, e.g., [17]),

$$\|\widetilde{u}_\perp\|_{L^\infty} \le C\|u_\perp\|^2_{H^{1/2}(\mathbb{R}^d)} \tag{7.15}$$

(here the fact that $u_\perp$ belongs to the subspace $H^{l/2}_\perp(\mathbb{R}^d)$ is, of course, irrelevant to the validity of (7.15); the same estimate holds for any $u\in H^{l/2}(\mathbb{R}^d)$). Thus, we have arrived at the VQ (7.2) with $\delta=0$; by Proposition 7.1,

$$\lim_{s\to 0} s^{\varkappa} n(s,(7.13)) = 0, \quad \forall\varkappa > \frac{d-1}{l-1}. \tag{7.16}$$

7. Finally, consider the VQ, generated by the second integral in the r.h.s. of (7.7):

$$\frac{v.p.\int_{|p^2-\lambda|<\varepsilon}(p^2-\lambda)^{-1}|u_\perp(p)|^2dp}{\|u\|^2_{H^{l/2}}}, \quad u_0\in H^{l/2}(\mathbb{R}^d). \tag{7.17}$$

As above, it reduces to the same VQ in the space $H^{l/2}_\perp(\mathbb{R}^d)$. Again using the notation (7.14) and estimating the integral in the numerator of (7.17), we get

$$v.p.\int_{|p^2-\lambda|<\varepsilon}(p^2-\lambda)^{-1}|u_\perp(p)|^2dp = \int_{\sqrt{\lambda-\varepsilon}}^{\sqrt{\lambda+\varepsilon}}(r^2-\lambda)^{-1}\widetilde{u}_\perp(r)dr \le C\|\widetilde{u}_\perp\|_{C^{\delta_0}}, \tag{7.18}$$

where the norm of $\widetilde{u}_\perp$ is taken in the Hölder space $C^{\delta_0}(\sqrt{\lambda-\varepsilon},\sqrt{\lambda+\varepsilon})$, $\delta_0>0$ — any number. The following estimate holds true for all $\delta>\delta_0$:

$$\|\widetilde{u}_\perp\|_{C^{\delta_0}} \le C\|u_\perp\|^2_{H^{\frac{1}{2}+\delta}}. \tag{7.19}$$

This estimate is well known to specialists in function theory; however, we could not find it in textbooks on Sobolev spaces. Using the standard technique of local change of coordinates, (7.19) can be reduced to a similar estimate with a hyperplane instead of the sphere S_λ; in the latter case, the proof of (7.19) is a matter of a simple calculation (similar to the one used in the proof of the embedding theorem (7.15)). The fact that $u_\perp$ belongs to the subspace $H_\perp^{l/2}(\mathbb{R}^d)$ is irrelevant to the validity of (7.19).

We use (7.19) and choose δ and δ_0 in such a way that $\frac{d-1}{l-2\delta-1} < d/l$. Thus, we arrive at the VQ (7.2) and so by Proposition 7.1,

$$\lim_{s\to 0} s^{\varkappa} n(s,(7.17)) = 0, \quad \forall \varkappa > \frac{d-1}{l-2\delta-1}. \tag{7.20}$$

8. The relations (7.7), (7.10), (7.12), (7.16), (7.20) yield

$$\lim_{s\to 0} s^{d/l} n(s,(5.4)) \le C_{5.9} + C\tau^{d/l};$$

since $\tau > 0$ can be taken arbitrary small, we get (5.9). ∎

9. Remarks on the proof of Theorem 1.2. In order to prove Theorem 1.2, one needs to make only minor modifications in the above construction. Let us briefly comment on this point. The representation (2.3) is still valid due to the abstract result [18, Theorem 1.2] (here the condition (1.14) is important). Next, the proof again reduces to the verification of the relations (2.5) (with $C_{1.15}$ instead of $C_{1.9}$) and (2.6). Further, in the same way the above question reduces to the consideration of variational quotients. A straightforward calculation shows that

$$\begin{aligned}
\operatorname{Im}(h(p^2)-\lambda-i0)^{-1} &= \frac{1}{h'(h^{-1}(\lambda))}\operatorname{Im}(p^2-h^{-1}(\lambda)-i0)^{-1},\\
\operatorname{Re}(h(p^2)-\lambda-i0)^{-1} &= F_\lambda(p^2)\operatorname{Re}(p^2-h^{-1}(\lambda)-i0)^{-1},
\end{aligned}$$

where the function

$$F_\lambda(x) := \frac{x-h^{-1}(\lambda)}{h(x)-\lambda}$$

is positive, C^1-smooth and bounded away from zero in the neighbourhood of $h^{-1}(\lambda)$. Thus, we get extra factors in the numerators of the VQ (5.4) and (5.6).

The extra factor in (5.6) does not affect further considerations in any way. The extra factor in (5.4) results in extra factors in the integrands in the numerators of (7.8), (7.11), (7.13), and (7.17). This will only affect constants in the estimates. Most importantly, one has to use the Hölder continuity of F_λ when proving the analogue of (7.18).

8 Proof of Proposition 7.1

The idea of the proof of Proposition 7.1 is to reduce the VQ (7.2) to a VQ on a sphere similarly to the construction of Section 6. However, here the technical details are more complicated.

1. *Change of function in* (7.2). The condition

$$u_\perp \perp H_0^{l/2}(S_\lambda^c) \quad \text{(orthogonality in } H^{l/2}) \tag{8.1}$$

can be stated as

$$((1-\triangle)^{l/4}u_\perp, (1-\triangle)^{l/4}\varphi)_{L^2} = 0, \quad \forall \varphi \in C_0^\infty(S_\lambda^c).$$

Write $l/2 = m - \theta$, where $m \in \mathbb{Z}$, $m - 1 < l/2 \le m$. Denote $v = (1-\triangle)^{-\theta} u_\perp$, $v \in H^{(l/2)+2\theta}(\mathbb{R}^d)$. Then the above condition can be stated in terms of v as

$$((1-\triangle)^m (1-\triangle)^{-l/4} v, (1-\triangle)^{l/4} \varphi)_{L^2} = 0, \quad \forall \varphi \in C_0^\infty(S_\lambda^c),$$

which is equivalent to the differential equation (m — integer!)

$$(1-\triangle)^m v(x) = 0, \quad x \in S_\lambda^c.$$

Thus, the spectrum of (7.2) coincides with the spectrum of the VQ

$$\frac{\|v\|^2_{H^{\frac{1}{2}+\delta+2\theta}(\mathbb{R}^d)}}{\|v\|^2_{H^{(l/2)+2\theta}(\mathbb{R}^d)}}, \quad v \in H^{(l/2)+2\theta}(\mathbb{R}^d), \quad (1-\triangle)^m v(x) = 0, \quad x \in S_\lambda^c. \tag{8.2}$$

The implicit orthogonality condition (8.1) has been reduced to an elliptic differential equation on v. Obviously, if l happens to be an integer even number, the above change of function becomes vacuous: in this case, $\theta = 0$, $m = l/2$ and $v = u_\perp$.

For $d = 1$, the solutions to the above differential equation form a *finite dimensional* space, and Proposition 7.1 follows immediately. Thus, below we consider the case $d \ge 2$.

2. *Separation of the interior and exterior.* Here our aim is to 'split' the VQ (8.2) into two VQ's: one in S_λ^i and another in S_λ^e. Below we are dealing with the Sobolev spaces $H^s(S_\lambda^i)$ and $H^s(S_\lambda^e)$, $s > 0$; the exact choice of one of the possible equivalent norms in these spaces will not be important for our construction. Denote by P_i and P_e the restriction operators

$$P_i : H^s(\mathbb{R}^d) \to H^s(S_\lambda^i), \quad u \mapsto u\,|_{S_\lambda^i}, \quad s > 0,$$
$$P_e : H^s(\mathbb{R}^d) \to H^s(S_\lambda^e), \quad u \mapsto u\,|_{S_\lambda^e}, \quad s > 0.$$

The operators P_i and P_e are bounded for any $s > 0$ and thus

$$\|P_i v\|^2_{H^{(l/2)+2\theta}(S_\lambda^i)} + \|P_e v\|^2_{H^{(l/2)+2\theta}(S_\lambda^e)} \le C \|v\|^2_{H^{(l/2)+2\theta}(\mathbb{R}^d)} \tag{8.3}$$

for any $v \in H^{(l/2)+2\theta}(\mathbb{R}^d)$. This estimate will allow us to 'split' the denominator of (8.2).

In order to 'split' the numerator, we will also need the bound of the opposite sign:

Proposition 8.1. *For any $s > 1/2$, $s - (1/2) \notin \mathbb{Z}$, there exists a constant $C_{8.4}$ such that for any $u \in H^s(\mathbb{R}^d)$,*

$$\|v\|^2_{H^s(\mathbb{R}^d)} \le C_{8.4}(\|P_i v\|^2_{H^s(S_\lambda^i)} + \|P_e v\|^2_{H^s(S_\lambda^e)}). \tag{8.4}$$

This statement is probably known to specialists, but we were unable to find it in the standard monographs on Sobolev spaces, so below we give a naive do-it-yourself proof. The assumption $s > 1/2$ is not necessary for the validity of (8.4), but sufficient for our purposes.

Obviously, for an integer s, (8.4) becomes a trivial equality with $C_{8.4} = 1$ (for the standard choice of norm in the Sobolev spaces). Thus, if the order $1/2 + \delta + 2\theta$ in the numerator of the VQ (8.2) is integer, we do not need to use the estimate (8.4). By using certain freedom of choice of the parameters m, θ and δ, we could make $1/2 + \delta + 2\theta$ equal to 1 when d is odd and to 0 when d is even. However, because of various restrictions on the parameters δ and θ, this route would allow us to cover only the case $l > d + 1/2$. We omit the details of this calculation.

Proof. 1. Let $\pi_e : H^s(S_\lambda^e) \to H^s(\mathbb{R}^d)$ be a bounded extension operator, $P_e\pi_e = \mathrm{id}$. Take $v_e = \pi_e P_e v$. Then

$$\|v_e\|_{H^s(\mathbb{R}^d)} \leq C\|P_e v\|_{H^s(S_\lambda^e)}. \tag{8.5}$$

2. Consider the function $w := P_i v - P_i v_e$. The function w satisfies the following boundary conditions on S_λ:

$$\frac{\partial^j w}{\partial |x|^j} \mid_{S_\lambda} = 0, \quad 0 \leq j < s - \frac{1}{2}.$$

Thus (see [17], Chapter 1, Theorem 11.5), w belongs to $H_0^s(S_\lambda^i)$. The extension by zero onto S_λ^e is a bounded operator from $H_0^s(S_\lambda^i)$ to $H^s(\mathbb{R}^d)$ (here the condition $s - (1/2) \notin \mathbb{Z}$ is important — see [17], Chapter 1, Theorem 11.4). Thus, denoting by $\widetilde{w}$ the extension of w by zero, we get

$$\begin{aligned}\|\widetilde{w}\|_{H^s(\mathbb{R}^d)} &\leq C\|w\|_{H^s(S_\lambda^i)} \leq C\|P_i v\|_{H^s(S_\lambda^i)} + C_1\|P_i v_e\|_{H^s(S_\lambda^i)} \\ &\leq C\|P_i v\|_{H^s(S_\lambda^i)} + C_2\|v_e\|_{H^s(S_\lambda^i)} \leq C\|P_i v\|_{H^s(S_\lambda^i)} + C_3\|P_e v\|_{H^s(S_\lambda^e)}, \end{aligned} \tag{8.6}$$

where we have used (8.5) on the last step.

3. Finally, note that $v = \widetilde{w} + v_e$. This is true on S_λ^i and S_λ^e by the definition of v_e and w and thus is also true on S_λ by the embedding theorem (recall that $s > 1/2$). Combining (8.5) and (8.6), we get (8.4). ∎

Now without the loss of generality assume that $\delta + 2\theta \notin \mathbb{Z}$ (otherwise we can slightly increase δ). Thus, by Proposition 8.1, for any $v \in H^{(1/2)+\delta+2\theta}(\mathbb{R}^d)$,

$$\|v\|^2_{H^{(1/2)+\delta+2\theta}(\mathbb{R}^d)} \leq C(\|P_i v\|^2_{H^{(1/2)+\delta+2\theta}(S_\lambda^i)} + \|P_e v\|^2_{H^{(1/2)+\delta+2\theta}(S_\lambda^e)}). \tag{8.7}$$

By (8.3) and (8.7), Proposition 7.1 reduces to the consideration of the two VQ,

$$\frac{\|v\|^2_{H^{(1/2)+\delta+2\theta}(S_\lambda^i)}}{\|v\|^2_{H^{(l/2)+2\theta}(S_\lambda^i)}}, \quad v \in H^{(l/2)+2\theta}(S_\lambda^i), \quad (1-\triangle)^m v(x) = 0, \tag{8.8}$$

$$\frac{\|v\|^2_{H^{(1/2)+\delta+2\theta}(S_\lambda^e)}}{\|v\|^2_{H^{(l/2)+2\theta}(S_\lambda^e)}}, \quad v \in H^{(l/2)+2\theta}(S_\lambda^e), \quad (1-\triangle)^m v(x) = 0. \tag{8.9}$$

It remains to prove the bound of the type (7.3) for the VQ (8.8) and (8.9).

3. *Reduction of* (8.8) *to a VQ on* S_λ. For $s \geq 0$ denote

$$H^{(s)}(S_\lambda) = \oplus_{j=0}^{m-1} H^{s-j}(S_\lambda); \tag{8.10}$$

let Γ_λ be the restriction operator,

$$\Gamma_\lambda : H^s(S_\lambda^i) \to H^{(s-\frac{1}{2})}(S_\lambda), \qquad v \mapsto v\mid_{S_\lambda} \oplus \tfrac{\partial}{\partial |x|} v\mid_{S_\lambda} \oplus \cdots \oplus \left(\tfrac{\partial}{\partial |x|}\right)^{m-1} v\mid_{S_\lambda}. \tag{8.11}$$

For any $s \in (0, 2m)$ and any v, $(1-\triangle)^m v = 0$, one has[1]

$$c\|v\|_{H^s(S_\lambda^i)} \leq \|\Gamma_\lambda v\|_{H^{(s-\frac{1}{2})}(S_\lambda)} \leq C\|v\|_{H^s(S_\lambda^i)}, \tag{8.12}$$

[1] Note that the delicate feature of this estimate is the range of the exponent s. For $s \geq 2m$, the estimate (8.12) is well known and can be found in a number of standard reference texts. However, for $s \in (0, 2m)$, the book [17] is the only source of reference we were able to find.

see, e.g., [17], Chapter 2, Theorem 7.4. Thus,

$$\frac{\|v\|^2_{H^{(1/2)+\delta+2\theta}(S^i_\lambda)}}{\|v\|^2_{H^{(l/2)+2\theta}(S^i_\lambda)}} \leq C\frac{\|\Gamma_\lambda v\|^2_{H^{(\delta+2\theta)}(S_\lambda)}}{\|\Gamma_\lambda v\|^2_{H^{(\frac{l-1}{2}+2\theta)}(S_\lambda)}},$$

and so it is sufficient to prove that for the VQ

$$\frac{\|w\|^2_{H^{(\delta+2\theta)}(S_\lambda)}}{\|w\|^2_{H^{(\frac{l-1}{2}+2\theta)}(S_\lambda)}}, \quad w \in H^{(\frac{l-1}{2}+2\theta)}(S_\lambda), \tag{8.13}$$

one has

$$\lim_{s\to 0} s^{\varkappa} n(s,(8.13)) < \infty, \quad \varkappa = \frac{d-1}{l-2\delta-1}. \tag{8.14}$$

Consider the VQ's

$$\frac{\|w\|^2_{H^{\delta+2\theta-j}(S_\lambda)}}{\|w\|^2_{H^{\frac{l-1}{2}+2\theta-j}(S_\lambda)}}, \quad w \in H^{\frac{l-1}{2}+2\theta-j}(S_\lambda), \tag{8.15j}$$

for $j = 0, \dots, m-1$. One has

$$n(s,(8.13)) \leq \sum_{j=0}^{m-1} n(s,(8.15j)).$$

Applying Proposition 6.1 to each of the VQ's (8.15j), we see that

$$\lim_{s\to 0} s^{\varkappa} n(s,(8.8)) < \infty, \quad \varkappa = \frac{d-1}{l-2\delta-1}. \tag{8.16}$$

4. *Reduction of* (8.9) *to a VQ on $S_\lambda \cup S_{4\lambda}$.* It remains to prove the estimate of the type (8.16) for the VQ (8.9). We proceed similarly to the proof of (8.16), but the additional technical difficulty to overcome is that the domain S^e_λ is not bounded. Because of this difficulty, we reduce the VQ (8.9) to a VQ on the union of two spheres, $S_\lambda \cup S_{4\lambda}$ (rather than on a sphere S_λ, as it would seem natural to do). The second sphere $S_{4\lambda}$ is the price we have to pay for getting rid of the infinity. Our estimates below are by far not optimal, but they are sufficient for our purposes.

Let $H^{(s)}(S_\lambda \cup S_{4\lambda}) = H^{(s)}(S_\lambda) \oplus H^{(s)}(S_{4\lambda})$ and let us define the restriction operator $\Gamma : H^s(S^e_\lambda) \to H^{(s-\frac{1}{2})}(S_\lambda \cup S_{4\lambda})$ by $\Gamma = \Gamma_\lambda \oplus \Gamma_{4\lambda}$.

We need an analogue of the estimate (8.12).

Proposition 8.2. *For any $s \in (0, 2m)$ and any v, $(1-\triangle)^m v = 0$, one has*

$$c\|v\|_{H^s(S^e_\lambda)} \leq \|\Gamma v\|_{H^{(s-\frac{1}{2})}(S_\lambda \cup S_{4\lambda})} \leq C\|v\|_{H^s(S^e_\lambda)}. \tag{8.17}$$

Using Proposition 8.2, we proceed exactly as for the VQ (8.8) and reduce (8.9) to a 'direct sum' of VQ's on $S_\lambda \cup S_{4\lambda}$. After that, using Proposition 6.1, we get

$$\lim_{s\to 0} s^{\varkappa} n(s,(8.9)) < \infty, \quad \varkappa = \frac{d-1}{l-2\delta-1},$$

which completes the proof of Proposition 7.1.

5. Proof of Proposition 8.2: 1. First note that the same arguments as for (8.12), applied to the domain $S^e_\lambda \cap S^i_{4\lambda}$, lead to the estimate

$$c\|v\|_{H^s(S^e_\lambda \cap S^i_{4\lambda})} \leq \|\Gamma v\|_{H^{(s-\frac{1}{2})}(S_\lambda \cup S_{4\lambda})} \leq C\|v\|_{H^s(S^e_\lambda \cap S^i_{4\lambda})}$$

for any $v \in H^s(S^e_\lambda \cap S^i_{4\lambda})$ such that $(1-\triangle)^m v = 0$. Thus, we only need to prove that for any $v \in H^s(S^e_\lambda)$, $(1-\triangle)^m v = 0$, one has

$$c\|v\|_{H^s(S^e_\lambda)} \leq \|v\|_{H^s(S^e_\lambda \cap S^i_{4\lambda})} \leq C\|v\|_{H^s(S^e_\lambda)}. \tag{8.18}$$

The second estimate in (8.18) is obviously true. Thus, we only need to prove the first estimate in (8.18).

2. Fix a 'cutoff function' $\eta \in C_0^\infty(\mathbb{R}^d)$, $\operatorname{supp}\eta \subset S^i_{3\lambda}$, $\eta(x) = 1$ for $x \in S^i_{2\lambda}$. We will prove two estimates:

$$\|\eta v\|_{H^s(S^e_\lambda)} \leq C\|v\|_{H^s(S^e_\lambda \cap S^i_{4\lambda})}, \tag{8.19}$$

$$\|(1-\eta)v\|_{H^s(S^e_\lambda)} \leq C\|v\|_{H^s(S^e_\lambda \cap S^i_{4\lambda})}, \tag{8.20}$$

where $v \in H^s(S^e_\lambda)$, $(1-\triangle)^m v = 0$.

3. Let us prove (8.19). Since ηv vanishes near $S_{4\lambda}$, one has

$$\|\eta v\|_{H^s(S^e_\lambda)} \leq C\|\eta v\|_{H^s(S^e_\lambda \cap S^i_{4\lambda})},$$

cf. [17, Chapter 1, Theorem 11.4]. Next, multiplication by η is a bounded operator in $H^s(S^e_\lambda \cap S^i_{4\lambda})$:

$$\|\eta v\|_{H^s(S^e_\lambda \cap S^i_{4\lambda})} \leq C\|v\|_{H^s(S^e_\lambda \cap S^i_{4\lambda})},$$

so we get (8.19). Note that we have not used the equation $(1-\triangle)^m v = 0$.

4. Let us prove (8.20). Consider the function $(1-\eta)v$, defined on S^e_λ. Denote by $\widetilde{v}$ the extension of $(1-\eta)v$ by zero onto S^i_λ. Clearly,

$$\|(1-\eta)v\|_{H^s(S^e_\lambda)} \leq C\|\widetilde{v}\|_{H^s(\mathbb{R}^d)}. \tag{8.21}$$

A direct computation (which uses the equation $(1-\triangle)^m v = 0$) shows that

$$(1-\triangle)^m \widetilde{v} = \sum_{0\leq|j|\leq 2m-1} \eta_j D^j v, \tag{8.22}$$

where $j = (j_1, j_2, \dots j_d)$ is a multi-index, $|j| = j_1 + \cdots + j_d$, $D^j = \left(\frac{\partial}{\partial x_1}\right)^{j_1} \dots \left(\frac{\partial}{\partial x_d}\right)^{j_d}$, and $\eta_j \in C_0^\infty(S^e_{2\lambda} \cap S^i_{3\lambda})$. Below we will check the estimate

$$\Big\| \sum_{0\leq|j|\leq 2m-1} \eta_j D^j v \Big\|_{H^{s-2m}(\mathbb{R}^d)} \leq C\|v\|_{H^s(S^e_\lambda \cap S^i_{4\lambda})}. \tag{8.23}$$

Using (8.23) and (8.21), we get

$$\begin{aligned}\|(1-\eta)v\|_{H^s(S^e_\lambda)} \leq C\|\widetilde{v}\|_{H^s(\mathbb{R}^d)} &= C\Big\|(1-\triangle)^{-m} \sum_{0\leq|j|\leq 2m-1} \eta_j D^j v\Big\|_{H^s(\mathbb{R}^d)} \\ &\leq C_1\Big\| \sum_{0\leq|j|\leq 2m-1} \eta_j D^j v\Big\|_{H^{s-2m}(\mathbb{R}^d)} \leq C_2\|v\|_{H^s(S^e_\lambda \cap S^i_{4\lambda})},\end{aligned}$$

which proves (8.20).

5. It remains to prove (8.23). To this end, fix another cutoff function $\beta \in C_0^\infty(\mathbb{R}^d)$, $\operatorname{supp}\beta \subset S_\lambda^e \cap S_{4\lambda}^i$, $\beta(x) = 1$ for $x \in S_{2\lambda}^e \cap S_{3\lambda}^i$. One has

$$\sum_{0\le|j|\le 2m-1} \eta_j D^j v = \sum_{0\le|j|\le 2m-1} \eta_j D^j(\beta v).$$

Thus,

$$\|\sum_{0\le|j|\le 2m-1} \eta_j D^j v\|_{H^{s-2m}(\mathbb{R}^d)} \le C \sum_{0\le|j|\le 2m-1} \|D^j(\beta v)\|_{H^{s-2m}(\mathbb{R}^d)}$$
$$\le C_1 \sum_{0\le|j|\le 2m-1} \|\beta v\|_{H^{s-2m+|j|}(\mathbb{R}^d)} \le C_2\|\beta v\|_{H^s(\mathbb{R}^d)} \le C_3\|\beta v\|_{H^s(S_\lambda^e\cap S_{4\lambda}^i)} \le C_4\|v\|_{H^s(S_\lambda^e\cap S_{4\lambda}^i)}. \quad \blacksquare$$

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