WEAKLY REGULAR FLOQUET HAMILTONIANS WITH PURE POINT SPECTRUM

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Abstract. We study the Floquet Hamiltonian $-i\partial_t + H + V(\omega t)$, acting in $L^2([0,T],\mathcal{H},dt)$, as depending on the parameter $\omega = 2\pi/T$. We assume that the spectrum of H in $\mathcal{H}$ is discrete, $\mathrm{Spec}(H) = \{h_m\}_{m=1}^{\infty}$, but possibly degenerate, and that $t \mapsto V(t) \in \mathcal{B}(\mathcal{H})$ is a 2π-periodic function with values in the space of Hermitian operators on $\mathcal{H}$. Let $J > 0$ and set $\Omega_0 = [\frac{8}{9}J, \frac{9}{8}J]$. Suppose that for some $\sigma > 0$ it holds true that $\sum_{h_m > h_n} \mu_{mn}(h_m - h_n)^{-\sigma} < \infty$ where $\mu_{mn} = (\min\{M_m, M_n\})^{1/2} M_m M_n$ and M_m is the multiplicity of h_m. We show that in that case there exist a suitable norm to measure the regularity of V, denoted ϵ_V, and positive constants, $\epsilon_\star$ and $\delta_\star$, with the property: if $\epsilon_V < \epsilon_\star$ then there exists a measurable subset $\Omega_\infty \subset \Omega_0$ such that its Lebesgue measure fulfills $|\Omega_\infty| \geq |\Omega_0| - \delta_\star \epsilon_V$ and the Floquet Hamiltonian has a pure point spectrum for all $\omega \in \Omega_\infty$.

1. Introduction

The problem we address in this paper concerns spectral analysis of so called Floquet Hamiltonians. The study of stability of non autonomous quantum dynamical systems is an effective tool to understand most of quantum problems which involve a small number of particles. When these systems are time-periodic the spectral analysis of the evolution operator over one period can give a fairly good information on this stability, see e.g. [1]. In fact this type of result generalises the celebrated RAGE theorem concerned with time-independent systems (one can consult [2] for a summary). As shown in [3] and [4] the spectral analysis of the evolution operator over one period (so called monodromy operator or Floquet operator) is equivalent to the spectral analysis of the corresponding Floquet Hamiltonian (sometimes called operator of quasi-energy). This is also what we are aiming for in this article. More precisely, we analyse time-periodic quantum systems which are weakly regular in time and "space" in the sense of an appropriately chosen norm, and give sufficient conditions to insure that the Floquet Hamiltonians has a pure point spectrum.

Such a program is not new. In the pioneering work [5] Bellissard has considered the so called pulsed rotor which is analytic in time and space, using a KAM type algorithm. Then Combescure [6] was able to treat harmonic oscillators driven by sufficiently smooth perturbations by adapting to quantum mechanics the well known Nash-Moser trick (c.f. [7] and [8]). Later on these ideas have been extended to a wider class of systems in [9]; it was even possible to require no regularity in space by using the so called adiabatic regularisation, originally proposed in [10] and further extended in [11], [12]. However none of these papers can be considered as optimal in the sense of having found the minimal value of regularity in time below which the Floquet Hamiltonian ceases to be pure point.

Though it is impossible to mention all the relevant contributions to the study of stability of time-dependent quantum systems we would like to mention the following ones. Perturbation theory for a fixed eigenvalue has been extended, in [13], to Floquet Hamiltonians which generically have a dense point spectrum. Bounded quasi-periodic time dependent perturbations of two level systems are considered in [14] whereas the case of unbounded perturbation of one dimensional oscillators are studied in [15]. Averaging methods combined with KAM techniques were described in [16] and [17].

In the present paper we attempt to further improve the KAM algorithm, particularly having in mind more optimal assumptions as far as the regularity in time is concerned. As a thorough analysis of the algorithm has shown this is possible owing to the fact that the algorithm contains several free parameters (for example the choice of norms in auxiliary Banach spaces that are constructed during the algorithm) which may be adjusted. This type of improvements is also illustrated on an example following Theorem 1 in Section 2. A more detailed discussion of this topic is postponed to concluding remarks in Section 10.

Another generalisation is that in the present result (Theorem 1) we allow degenerate eigenvalues of the unperturbed Hamilton operator (denoted H in what follows). The degeneracy of eigenvalues h_m of H can grow arbitrarily fast with m provided the time-dependent perturbation is sufficiently regular. To our knowledge this is a new feature in this context. Previously two conditions were usually imposed, namely bounded degeneracy and a growing gap condition on eigenvalues h_m, reducing this way the scope of applications of this theory to one dimensional confined systems. Owing to the generalisation to degenerate eigenvalues we are able to consider also some models in higher dimensions, for example the N-dimensional quantum top, i.e., the N-dimensional version of the pulsed rotor. A short description of this model is given, too, in Section 2 after Theorem 1.

The article is organised as follows. In Section 2 we introduce the notation and formulate the main theorem. The basic idea of the KAM-type algorithm is outlined in Section 3. The algorithm consists in an iterative procedure resulting in diagonalisation of the Floquet Hamiltonian. For this sake one constructs an auxiliary sequence of Banach spaces which form in fact a directed sequence. The procedure itself may formally be formulated in terms of an inductive limit. Sections 4–8 contain some additional results needed for the proof, particularly the details of the construction of the auxiliary Banach spaces and how they are related to Hermitian operators in the given Hilbert space, and a construction of the set of "non-resonant" frequencies for which the Floquet Hamiltonian has a pure point spectrum (the frequency is considered as a parameter). Section 9 is devoted to the proof of Theorem 1. In Section 10 we conclude our presentation with several remarks concerning comparison of the result stated in Theorem 1 with some previous ones.

2. Main theorem

The central object we wish to study in this paper is a self-adjoint operator of the form $\mathbf{K}+\mathbf{V}$ acting in the Hilbert space

$$\mathcal{K} = L^2([\,0,T\,],\mathrm{dt})\otimes\mathcal{H}\cong L^2([\,0,T\,],\mathcal{H},\mathrm{dt})$$

where $T = 2\pi/\omega$, ω is a positive number (a frequency) and $\mathcal{H}$ is a fixed separable Hilbert space. The operator $\mathbf{K}$ is self-adjoint and has the form

$$\mathbf{K} = -i\,\partial_t\otimes 1 + 1\otimes H$$

where the differential operator $-i\partial_t$ acts in $L^2([\,0,T\,],\mathrm{dt})$ and represents the self-adjoint operator characterised by periodic boundary conditions. This means that the eigenvalues of $-i\partial_t$ are $k\omega$, $k\in\mathbb{Z}$, and the corresponding normalised eigenvectors are $\chi_k(t) = T^{-1/2}\exp(ik\omega t)$. H is a self-adjoint operator in $\mathcal{H}$ and is supposed to have a discrete spectrum. Finally, $\mathbf{V}$ is a bounded Hermitian operator in $\mathcal{K}$ determined by a measurable operator-valued function $t\mapsto V(\omega t)\in\mathcal{B}(\mathcal{H})$ such that $\sup_{t\in\mathbb{R}}\|V(t)\|<\infty$, $V(t)$ is 2π-periodic, and for almost all $t\in\mathbb{R}$, $V(t)^* = V(t)$. Naturally, $(\mathbf{V}\psi)(t) = V(\omega t)\psi(t)$ in $\mathcal{K}\cong L^2([\,0,T\,],\mathcal{H},\mathrm{dt})$.

Let

$$\sum_{k\in\mathbb{Z}} k\omega\,P_k$$

be the spectral decomposition of $-i\partial_t$ in $L^2([\,0,T\,],\mathrm{dt})$ and let

$$H = \sum_{m\in\mathbb{N}} h_m\,Q_m$$

be the spectral decomposition of H in $\mathcal{H}$. Thus we can write

$$\mathcal{H} = \sum_{m\in\mathbb{N}}^{\oplus}\mathcal{H}_m$$

where $\mathcal{H}_m = \operatorname{Ran} Q_m$ are the eigenspaces. We suppose that the multiplicities are finite,

$$M_m = \dim\mathcal{H}_m<\infty,\ \forall m\in\mathbb{N}.$$

Hence the spectrum of $\mathbf{K}$ is pure point and its spectral decomposition reads

$$\mathbf{K} = \sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{N}}(k\omega+h_m)\,P_k\otimes Q_m, \tag{1}$$

implying a decomposition of $\mathcal{K}$ into a direct sum,

$$\mathcal{K} = \sum_{(k,m)\in\mathbb{Z}\times\mathbb{N}}^{\oplus}\operatorname{Ran}(P_k\otimes Q_m)\,.$$

Here is some additional notation. Set

$$V_{knm} = \frac{1}{T}\int_0^T e^{-ik\omega t}\,Q_nV(\omega t)\,Q_m\,\mathrm{dt} = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}\,Q_nV(t)\,Q_m\,\mathrm{dt}\in\mathcal{B}(\mathcal{H}_m,\mathcal{H}_n)\,. \tag{2}$$

Further,

$$\Delta_{mn} = h_m - h_n,$$

and

$$\Delta_0 = \inf_{m\neq n} |\Delta_{mn}|.$$

Finally we set

$$\mu_{mn} = (\min\{M_m, M_n\})^{1/2} M_m M_n.$$

Now we are able to formulate our main result. Though not indicated explicitly in the notation the operator $\mathbf{K}+\mathbf{V}$ is considered as depending on the parameter ω.

Theorem 1. *Fix $J>0$ and set $\Omega_0 := [\,\frac{8}{9}J, \frac{9}{8}J\,]$. Assume that $\Delta_0>0$ and that there exists $\sigma>0$ such that*

$$\Delta_\sigma(J) := J^\sigma \sum_{\substack{m,n\in\mathbb{N}\\ \Delta_{mn}>J/2}} \frac{\mu_{mn}}{(\Delta_{mn})^\sigma} < \infty\,.$$

Then for every $r>\sigma+\frac{1}{2}$ there exist positive constants (depending, as indicated, on σ, r, Δ_0 and J but independent of V), $\epsilon_\star(r,\Delta_0,J)$ and $\delta_\star(\sigma,r,J)$, with the property: if

$$\epsilon_V := \sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{N}} \|V_{knm}\| \, \max\{|k|^r, 1\} < \min\left\{\epsilon_\star(r,\Delta_0,J), \frac{|\Omega_0|}{\delta_\star(\sigma,r,J)}\right\}$$

(here $|\Omega_*|$ stands for the Lebesgue measure of Ω_*) then there exists a measurable subset $\Omega_\infty\subset\Omega_0$ such that

$$|\Omega_\infty| \geq |\Omega_0| - \delta_\star(\sigma,r,J)\,\epsilon_V \tag{3}$$

and the operator $\mathbf{K}+\mathbf{V}$ has a pure point spectrum for all $\omega\in\Omega_\infty$

Remarks. 1) In the course of the proof we shall show even more. Namely, for all $\omega\in\Omega_\infty$ and any eigenvalue of $\mathbf{K}+\mathbf{V}$ the corresponding eigen-projector P belongs to the Banach algebra with the norm

$$\|P\| = \sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{N}} \|P_{knm}\| \, \max\{|k|^{r-\sigma-\frac{1}{2}}, 1\}.$$

This shows that P is $(r-\sigma-1/2)$-differentiable as a map from $[\,0,T\,]$ to the space of bounded operators in $\mathcal{H}$

2) The constants $\epsilon_\star(r,\Delta_0,J)$ and $\delta_\star(\sigma,r,J)$ are in fact known quite explicitly and are given by formulae (70), (71), (77) and (78). Setting $\alpha=2$ and $q^r=e^2$ in these formulae (this is a possible choice) we get

$$\epsilon_\star(r,\Delta_0,J) = \frac{\min\{4\Delta_0,\ J\}}{270\,e^3},$$

and

$$\begin{aligned}\delta_\star(\sigma,r,J) &= 1440\,e^5 2^\sigma \left(\frac{2\sigma+1}{\left(1-e^{-\frac{2}{r}}\right)e}\right)^{\sigma+\frac{1}{2}} \left(\sum_{s=1}^{\infty} s^2 e^{-\frac{2}{r}(r-\sigma-\frac{1}{2})s}\right) \Delta_\sigma(J)\\ &= 1440 \left(\frac{2\sigma+1}{\left(1-e^{-\frac{2}{r}}\right)e}\right)^{\sigma+\frac{1}{2}} 2^\sigma e^{3+\frac{2}{r}\left(\sigma+\frac{1}{2}\right)} \frac{1+e^{-2+\frac{2}{r}\left(\sigma+\frac{1}{2}\right)}}{\left(1-e^{-2+\frac{2}{r}\left(\sigma+\frac{1}{2}\right)}\right)^3} \Delta_\sigma(J)\end{aligned}$$

3) The formulae for $\epsilon_\star$ and $\delta_\star$ can be further simplified if we assume that r is not too big, more precisely under the assumption that $r \le \frac{7}{8}(2\sigma+1)$ (if this is not the case we can always replace r by a smaller value but still requiring that $r > \sigma + \frac{1}{2}$). A better choice than that made in the previous remark is $\alpha = 2$ and $q = e^{4/(2\sigma+1)}$. We get (c.f. (71))

$$\epsilon_\star(r, \Delta_0, J) = \frac{\min\{4\,\Delta_0, J\}}{270\,e}\, e^{-4r/(2\sigma+1)} \ge \frac{\min\{4\,\Delta_0, J\}}{270\,e^{9/2}}$$

and (c.f. (77) and (78))

$$\delta_\star(\sigma, r, J) = 1440\,e\,2^\sigma \left(\frac{2\sigma+1}{\left(1-e^{-\frac{4}{2\sigma+1}}\right)e}\right)^{\sigma+\frac{1}{2}} e^{\frac{8r}{2\sigma+1}} \left(\sum_{s=1}^{\infty} s^2 e^{-2\frac{2r-2\sigma-1}{2\sigma+1}s}\right) \Delta_\sigma(J)\,.$$

Using the estimate

$$\sum_{s=1}^{\infty} s^2 e^{-2xs} = \frac{\cosh(x)}{4\,\sinh(x)^3} \le \frac{1}{4\,x^3}$$

we finally obtain

$$\delta_\star(\sigma, r, J) \;\le\; 45\,e\,2^\sigma \left(\frac{2\sigma+1}{\left(1-e^{-\frac{4}{2\sigma+1}}\right)e}\right)^{\sigma+\frac{1}{2}} e^{\frac{8r}{2\sigma+1}} \left(\frac{2\sigma+1}{r-\sigma-\frac{1}{2}}\right)^3 \Delta_\sigma(J)\,.$$

We conclude this section with a brief description of two models illustrating the effectiveness of Theorem 1. In the first model we set $\mathcal{H} = L^2([\,0,1\,], \mathrm{dx})$, $H = -\partial_x^2$ with Dirichlet boundary conditions, and $V(t) = z(t)x^2$ where $z(t)$ is a sufficiently regular 2π-periodic function. As shown in [18] the spectral analysis of this simple model is essentially equivalent to the analysis of the so called quantum Fermi accelerator. The particularity of the latter model is that the underlying Hilbert space itself is time-dependent, $\mathcal{H}_t = L^2([\,0, a(t)\,], \mathrm{dx})$ where $a(t)$ is a strictly positive periodic function. The time-dependent Hamiltonian is $-\partial_x^2$ with Dirichlet boundary conditions. Using a convenient transformation one can pass from the Fermi accelerator to the former model getting the function $z(t)$ expressed in terms of $a(t)$, $a'(t)$ and $a''(t)$. But let us return to the analysis of our model. Eigenvalues of H are non-degenerate, $h_m = m^2\pi^2$ for $m \in \mathbb{N}$, with normalised eigenfunctions equal to $\sqrt{2}\sin(m\pi x)$. Note that in the notation we are using in the present paper $0 \notin \mathbb{N}$. A straightforward calculation gives

$$V_{knm} = z_k \times \begin{cases} \frac{8(-1)^{m+n} mn}{(m^2-n^2)^2\pi^2} & \text{if } m \ne n, \\[2ex] \frac{1}{3} - \frac{1}{2m^2\pi^2} & \text{if } m = n, \end{cases}$$

where $z_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt} z(t)\,dt$ is the Fourier coefficient of $z(t)$. Hence one derives that

$$\epsilon_V = \sup_{n\in\mathbb{N}} \left(\frac{1}{3} + \frac{2}{n^2\pi^2} + \frac{4}{\pi^2}\sum_{j=1}^{n-1}\frac{1}{j^2}\right) \sum_{k\in\mathbb{Z}} |z_k| \max\{|k|^r, 1\} = \sum_{k\in\mathbb{Z}} |z_k| \max\{|k|^r, 1\}.$$

For any $J > 0$, $\Delta_\sigma(J)$ is finite if and only if $\sigma > 1$. On the other hand, to have ϵ_V finite it is sufficient that $z(t) \in C^s$ where $s > r + 1 > \sigma + \frac{1}{2} + 1 > \frac{5}{2}$. So $z(t) \in C^3$

suffices for the theory to be applicable. This may be compared to an older result in [9], §4.2, giving a much worse condition, namely $z(t) \in C^{17}$.
The second model is the pulsed rotator in N dimensions. In this case $\mathcal{H} = L^2(S^N, d\mu)$, with $S^N \subset \mathbb{R}^{N+1}$ being the N-dimensional unit sphere with the standard (rotationally invariant) Riemann metric and the induced normalised measure $d\mu$, and $H = -\Delta_{LB}$ is the Laplace-Beltrami operator on S^N. The spectrum of H is well known, $\mathrm{Spec}(H) = \{h_m\}_{m=0}^{\infty}$, where

$$h_m = m(m + N - 1)$$

and the multiplicities are

$$M_m = \binom{m+N}{N} - \binom{m+N-2}{N}.$$

The time-dependent operator $V(t)$ in $\mathcal{H}$ acts via multiplication, $(V(t)\varphi)(x) = v(t,x)\varphi(x)$, where $v(t,x)$ is a real measurable bounded function on $\mathbb{R} \times S^N$ which is 2π-periodic in the variable t. Consequently, $\mathcal{K} \cong L^2([\,0,T\,]\times S^N, dt\, d\mu)$ and $(\mathbf{V}\psi)(t,x) = v(\omega t, x)\psi(t,x)$. Note that the asymptotic behaviour of the eigenvalues and the multiplicities, as $m \to \infty$, is $h_m \sim m^2$, $M_m \sim (2/(N-1)!)\, m^{N-1}$. So $\Delta_\sigma(J)$ is finite, for any $J > 0$, if and only if

$$\sum_{m^2 - n^2 > J/2} \frac{n^{\frac{3}{2}(N-1)} m^{N-1}}{(m^2 - n^2)^\sigma} < \infty\,.$$

To ensure this condition we require that $\sigma > \frac{5}{2}(N-1) + 1$. Let us assume that there exist $s, u \in \mathbb{Z}_+$ such that, for any system of local (smooth) coordinates $(y_1, \dots, y_N)$ on S^N, the derivatives $\partial_t^\alpha \partial_{y_1}^{\beta_1} \dots \partial_{y_N}^{\beta_N} v(t, y_1, \dots, y_N)$ exist and are continuous for all α, β, $\alpha \le s$ and $\beta_1 + \ldots + \beta_N \le u$. If $u \ge 4$ then $[\,H, [\,H, V(t)\,]]$ is a well defined second order differential operator with continuous coefficient functions and the operator $[\,H, [\,H, V(t)\,]](1 + H)^{-1}$ is bounded. Clearly,

$$\frac{(h_m - h_n)^2}{1 + h_m}\, Q_n V(t) Q_m = Q_n [\,H, [\,H, V(t)\,]](1 + H)^{-1} Q_m.$$

Using this relation one derives an estimate on V_{knm},

$$\|V_{knm}\| \le \mathrm{const}\, \frac{1 + \min\{h_n, h_m\}}{|k|^s (h_m - h_n)^2}\,,$$

valid for $k \ne 0$ and $m \ne n$. The number

$$\sup_{n \in \mathbb{Z}_+} \sum_{m \in \mathbb{Z}_+, m \ne n} \frac{1 + \min\{h_n, h_m\}}{(h_m - h_n)^2}$$

is finite. To see it one can employ the asymptotics of h_m and the fact that the sequence

$$a_n = \sum_{m \in \mathbb{Z}_+, m \ne n} \frac{1 + \min\{n^2, m^2\}}{(m^2 - n^2)^2} = \left(1 + \frac{1}{n^2}\right)\frac{\pi^2}{12} - \frac{3}{16n^2} + \frac{5}{16n^4} - \frac{1}{2n}\sum_{m=1}^{2n-1} \frac{1}{m}\,,$$

$n = 1, 2, 3, \ldots$, is bounded. It follows that the norm ϵ_V is finite if $s > r+1 > \sigma + \frac{1}{2} + 1 > \frac{5}{2}(N-1) + 1 + \frac{3}{2} = \frac{5}{2}\,N$. Thus the theory is applicable provided $u \ge 4$ and $s > \frac{5}{2}N$. The same example has also been treated by adiabatic methods in [11]. In that case the assumptions are weaker. It suffices that $v(t,x)$ be $(N+1)$-times differentiable in t with all derivatives $\partial_t^\alpha v(t,x)$, $0 \le \alpha \le N+1$, uniformly bounded. However the conclusion is somewhat weaker as well. Under this assumption $\mathbf{K}+\mathbf{V}$ has no absolutely continuous spectrum but nothing is claimed about the singular continuous spectrum.

3. Formal limit procedure

Suppose there is given a directed sequence of real or complex Banach spaces, $\{\mathfrak{X}_s\}_{s=0}^\infty$, with linear mappings

$$\iota_{us} : \mathfrak{X}_s \to \mathfrak{X}_u \quad \text{if } s \le u, \text{ with } \|\iota_{us}\| \le 1,$$

(and ι_{ss} is the unite mapping in $\mathfrak{X}_s$) and such that

$$\iota_{vu}\iota_{us} = \iota_{vs} \quad \text{if } s \le u \le v\,.$$

To simplify the notation we set in what follows

$$\iota_s = \iota_{s+1,s}.$$

Denote by $\mathfrak{X}_\infty$ the norm inductive limit of $\{\mathfrak{X}_s,\ \iota_{us}\}$ in the sense of [19], §1.3.4 or [20], §1.23 (the algebraic inductive limit is endowed with a seminorm induced by $\limsup_s \|\cdot\|_s$, the kernel of this seminorm is divided out and the result is completed). $\mathfrak{X}_\infty$ is related to the original directed sequence via the mappings $\iota_{\infty s} : \mathfrak{X}_s \to \mathfrak{X}_\infty$ obeying $\|\iota_{\infty s}\| \le 1$ and $\iota_{\infty u}\iota_{us} = \iota_{\infty s}$ if $s \le u$. By the construction, the union $\bigcup_{s \ge s_0} \iota_{\infty s}(\mathfrak{X}_s)$ is dense in $\mathfrak{X}_\infty$ for any $s_0 \in \mathbb{Z}_+$.
If $\{A_s \in \mathcal{B}(\mathfrak{X}_s)\}$ is a family of bounded operators, defined for $s \ge s_0$ and such that

$$A_u\iota_{us} = \iota_{us}A_s \quad \text{if } s_0 \le s \le u, \text{ and } \sup_s \|A_s\| < \infty,$$

then $A_\infty \in \mathcal{B}(\mathfrak{X}_\infty)$ designates the inductive limit of this family characterised by the property $A_\infty\iota_{\infty s} = \iota_{\infty s}A_s$, $\forall s \ge s_0$.
Let $\mathcal{D}_\infty \in \mathcal{B}(\mathfrak{X}_\infty)$ be the inductive limit of a family of bounded operators $\{\mathcal{D}_s \in \mathcal{B}(\mathfrak{X}_s);\ s \ge 0\}$, with the property

$$\|\mathcal{D}_s\| \le 1,\ \|1-\mathcal{D}_s\| \le 1, \quad \forall s. \tag{4}$$

We also suppose that there is given a sequence of one-dimensional spaces $\mathbf{k}K_s$, $s = 0,1,\ldots,\infty$, where the K_s are distinguished basis elements. Here the field $\mathbf{k}$ is either $\mathbb{C}$ or $\mathbb{R}$ depending on whether the Banach spaces $\mathfrak{X}_s$ are complex or real. Set

$$\tilde{\mathfrak{X}}_s = \mathbf{k}K_s \oplus \mathfrak{X}_s, \quad s = 0,1,\ldots,\infty.$$

Then $\{\tilde{\mathfrak{X}}_s\}_{s=0}^\infty$ becomes a directed sequence of vector spaces provided one defines $\tilde{\iota}_{us} : \tilde{\mathfrak{X}}_s \to \tilde{\mathfrak{X}}_u$ by

$$\tilde{\iota}_{us}|_{\mathfrak{X}_s} = \iota_{us} \text{ and } \tilde{\iota}_{us}(K_s) = K_u \quad \text{if } s \le u.$$

Set

$$\phi(x) = \frac{1}{x}\left(e^x - \frac{e^x-1}{x}\right) = \sum_{k=0}^\infty \frac{k+1}{(k+2)!}\,x^k\,. \tag{5}$$

Proposition 2. *Suppose that, in addition to the sequences* $\{\mathfrak{X}_s\}_{s=0}^{\infty}$, $\{K_s\}_{s=0}^{\infty}$ *and* $\{\mathcal{D}_s\}_{s=0}^{\infty}$, *there are given sequences* $\{V_s\}_{s=0}^{\infty}$ *and* $\{\Theta_u^s\}_{u=s+1}^{\infty}$ *such that* $V_s \in \mathfrak{X}_s$, $\Theta_u^s \in \mathcal{B}(\mathfrak{X}_u)$, *and*

$$\Theta_v^s \iota_{vu} = \iota_{vu} \Theta_u^s \quad \textit{if } s < u \le v. \tag{6}$$

Set

$$T_s = e^{\Theta_s^{s-1}} e^{\Theta_s^{s-2}} \dots e^{\Theta_s^0} \in \mathcal{B}(\mathfrak{X}_s) \quad \textit{for } s \ge 1\,. \tag{7}$$

Let $\{W_s\}_{s=0}^{\infty}$ *be another sequence, with* $W_s \in \mathfrak{X}_s$, *defined recursively:*

$$\begin{aligned} W_0 &= V_0, \\ W_{s+1} &= \iota_s(W_s) + T_{s+1}(V_{s+1} - \iota_s(V_s)) \\ &\quad + \Theta_{s+1}^s \phi(\Theta_{s+1}^s) \iota_s (1 - \mathcal{D}_s)(W_s - \iota_{s-1}(W_{s-1})), \end{aligned} \tag{8}$$

where we set, by convention, $\mathfrak{X}_{-1} = 0$, $W_{-1} = 0$. *Extend the mappings* Θ_u^s *to* $\tilde{\Theta}_u^s : \tilde{\mathfrak{X}}_u \to \tilde{\mathfrak{X}}_u$ *by*

$$\tilde{\Theta}_u^s(K_u) = -\Theta_u^s \mathcal{D}_u(\iota_{us}(W_s)) - (1 - \mathcal{D}_u)(\iota_{us}(W_s) - \iota_{u,s-1}(W_{s-1})), \tag{9}$$

and consequently the mappings T_s *to* $\tilde{T}_s : \tilde{\mathfrak{X}}_s \to \tilde{\mathfrak{X}}_s$,

$$\tilde{T}_s = e^{\tilde{\Theta}_s^{s-1}} e^{\tilde{\Theta}_s^{s-2}} \dots e^{\tilde{\Theta}_s^0} \; \textit{for } s \ge 1, \; \tilde{T}_0 = 1.$$

Then it holds

$$\tilde{T}_s(K_s + V_s) = K_s + \mathcal{D}_s(W_s) + (1 - \mathcal{D}_s)(W_s - \iota_{s-1}(W_{s-1})), \quad s = 0, 1, 2, \dots. \tag{10}$$

Remark. Since $\tilde{\Theta}_u^s(K_u) \in \mathfrak{X}_u$ it is easy to observe that

$$\tilde{T}_s(K_s) - K_s \in \mathfrak{X}_s.$$

Furthermore, note that (9) implies that $\tilde{\Theta}_v^s(K_v) = \iota_{vu}\tilde{\Theta}_u^s(K_u)$ if $0 \le s < u \le v$, and so the mappings $\tilde{\Theta}_u^s$ still satisfy

$$\tilde{\Theta}_v^s \tilde{\iota}_{vu} = \tilde{\iota}_{vu} \tilde{\Theta}_u^s \quad \text{if } s < u \le v.$$

Proof. By induction in s. For $s = 0$ the claim is obvious. In the induction step $s \to s+1$ one may use the induction hypothesis and relations (9) and (8):

$$\begin{aligned} \tilde{T}_{s+1}(K_{s+1} + V_{s+1}) &= \tilde{T}_{s+1}\tilde{\iota}_s(K_s + V_s) + T_{s+1}(V_{s+1} - \iota_s(V_s)) \\ &= e^{\tilde{\Theta}_{s+1}^s} \tilde{\iota}_s \tilde{T}_s(K_s + V_s) + T_{s+1}(V_{s+1} - \iota_s(V_s)) \\ &= e^{\tilde{\Theta}_{s+1}^s} \tilde{\iota}_s(K_s + \mathcal{D}_s(W_s) + (1 - \mathcal{D}_s)(W_s - \iota_{s-1}(W_{s-1}))) \\ &\quad + T_{s+1}(V_{s+1} - \iota_s(V_s)) \\ &= K_{s+1} + \mathcal{D}_{s+1}(\iota_s(W_s)) + \frac{e^{\Theta_{s+1}^s} - 1}{\Theta_{s+1}^s} \tilde{\Theta}_{s+1}^s \tilde{\iota}_s(K_s + \mathcal{D}_s(W_s)) \\ &\quad + e^{\Theta_{s+1}^s} \iota_s(1 - \mathcal{D}_s)(W_s - \iota_{s-1}(W_{s-1})) + T_{s+1}(V_{s+1} - \iota_s(V_s)) \\ &= K_{s+1} - (1 - \mathcal{D}_{s+1})\iota_s(W_s) + \iota_s(W_s) + T_{s+1}(V_{s+1} - \iota_s(V_s)) \\ &\quad + \left(e^{\Theta_{s+1}^s} - \frac{e^{\Theta_{s+1}^s} - 1}{\Theta_{s+1}^s} \right) \iota_s(1 - \mathcal{D}_s)(W_s - \iota_{s-1}(W_{s-1})) \\ &= K_{s+1} - (1 - \mathcal{D}_{s+1})\iota_s(W_s) + W_{s+1} \\ &= K_{s+1} + \mathcal{D}_{s+1}(W_{s+1}) + (1 - \mathcal{D}_{s+1})(W_{s+1} - \iota_s(W_s))\,. \end{aligned}$$

□

Proposition 3. *Assume that the sequences* $\{V_s\}_{s=0}^{\infty}$, $\{W_s\}_{s=0}^{\infty}$ *and* $\{\Theta_u^s\}_{u=s}^{\infty}$ *have the same meaning and obey the same assumptions as in Proposition 2. Denote*

$$w_s = \|W_s - \iota_{s-1}(W_{s-1})\|$$

(with $w_0 = \|W_0\|$*). Assume, in addition, that there exist a sequence of positive real numbers,* $\{F_s\}_{s=0}^{\infty}$, *such that*

$$\|\Theta_u^s\| \leq F_s w_s, \quad \forall s, u, \ u > s, \tag{11}$$

a sequence of non-negative real numbers $\{v_s\}_{s=0}^{\infty}$ *such that*

$$\|V_s - \iota_{s-1}(V_{s-1})\| \leq v_s, \ \forall s,$$

(for $s = 0$ *this means* $\|V_0\| \leq v_0$*) and a constant* $A \geq 0$ *such that*

$$F_s v_s^2 \leq A\, v_{s+1}, \ \forall s, \tag{12}$$

and that it holds true

$$B = \sum_{s=0}^{\infty} F_s v_s < \infty. \tag{13}$$

Denote

$$C = \sup_s F_s v_s. \tag{14}$$

If $d > 0$ *obeys*

$$e^{dB} + A\phi(dC)\, d^2 \leq d \tag{15}$$

then

$$w_s \leq d\, v_s, \ \forall s. \tag{16}$$

Proof. We shall proceed by induction in s. If $s = 0$ then $v_0 = w_0 = \|V_0\|$ and (16) holds true since (15) implies that $d \geq 1$. The induction step $s \to s+1$: according to (8), (7), (4) and (15), and owing to the fact that $\phi(x)$ is monotone, we have

$$\begin{aligned}
w_{s+1} &\leq \|T_{s+1}\|\, v_{s+1} + \|\Theta_{s+1}^s\|\, \phi(\|\Theta_{s+1}^s\|)\, w_s \\
&\leq \exp\left(\sum_{j=0}^{s} F_j w_j\right) v_{s+1} + \phi(F_s w_s) F_s w_s^2 \\
&\leq \exp\left(d \sum_{j=0}^{s} F_j v_j\right) v_{s+1} + \phi(d\, F_s v_s) F_s d^2 v_s^2 \\
&\leq e^{dB} v_{s+1} + \phi(dC) d^2 A\, v_{s+1} \\
&\leq d\, v_{s+1}.
\end{aligned}$$

□

Remark. If

$$B \leq \frac{1}{3}\ln 2 \quad \text{and} \quad A\,\phi(3C) \leq \frac{1}{9}$$

then (15) holds true with $d = 3$.

Recall that $\Theta_\infty^s \in \mathcal{B}(\mathfrak{X}_\infty)$ is the unique bounded operator on $\mathfrak{X}_\infty$ such that

$$\Theta_\infty^s \iota_{\infty u} = \iota_{\infty u} \Theta_u^s, \ \forall u > s.$$

If (11) is true then its norm is estimated by

$$\|\Theta_\infty^s\| \leq F_s w_s. \tag{17}$$

Corollary 4. *Under the same assumptions as in Proposition 3, if $d > 0$ exists such that condition (15) is satisfied, and*

$$F_{\inf} = \inf_s F_s > 0 \tag{18}$$

then the limits

$$V_\infty = \lim_{s\to\infty} \iota_{\infty s}(V_s), \quad W_\infty = \lim_{s\to\infty} \iota_{\infty s}(W_s)$$

exist in $\mathfrak{X}_\infty$, the limit

$$T_\infty = \lim_{s\to\infty} e^{\Theta_\infty^{s-1}} \ldots e^{\Theta_\infty^0}$$

exists in $\mathcal{B}(\mathfrak{X}_\infty)$, and $T_\infty \in \mathcal{B}(\mathfrak{X}_\infty)$ can be extended to a linear mapping $\tilde{T}_\infty : \tilde{\mathfrak{X}}_\infty \to \tilde{\mathfrak{X}}_\infty$ by

$$\tilde{T}_\infty(K_\infty) - K_\infty = \lim_{s\to\infty} \iota_{\infty s}\left(\tilde{T}_s(K_s) - K_s\right), \tag{19}$$

with the limit existing in $\mathfrak{X}_\infty$. These objects obey the equality

$$\tilde{T}_\infty(K_\infty + V_\infty) = K_\infty + \mathcal{D}_\infty(W_\infty). \tag{20}$$

Proof. If $u \geq s$ then

$$\|\iota_{\infty u}(V_u) - \iota_{\infty s}(V_s)\| \ = \ \Big\| \sum_{j=s+1}^{u} \iota_{\infty j}(V_j - \iota_{j-1}(V_{j-1}))\Big\| \leq \sum_{j=s+1}^{u} v_j.$$

Since

$$\sum_{s=0}^{\infty} v_s \leq \frac{1}{F_{\inf}} \sum_{s=0}^{\infty} F_s v_s < \infty$$

the sequence $\{\iota_{\infty s}(V_s)\}$ is Cauchy in $\mathfrak{X}_\infty$ and so $V_\infty \in \mathfrak{X}_\infty$ exists. Under assumption (16) we can apply the same reasoning to the sequence $\{\iota_{\infty s}(W_s)\}$ to conclude that the limit $W_\infty = \lim_{s\to\infty} \iota_{\infty s}(W_s)$ exists in $\mathfrak{X}_\infty$. Set

$$\bar{T}_s = e^{\Theta_\infty^{s-1}} \ldots e^{\Theta_\infty^0} \quad \text{if } s \geq 1, \text{ and } \bar{T}_0 = 1.$$

If $u \geq s$ then, owing to (17) and (16), we have

$$\begin{aligned} \|\bar{T}_u - \bar{T}_s\| \ &\leq \ \left(\exp\left(\sum_{j=s}^{u-1} \|\Theta_\infty^j\|\right) - 1\right) \exp\left(\sum_{j=0}^{s-1} \|\Theta_\infty^j\|\right) \\ &\leq \ \exp\left(d\sum_{j=0}^{u-1} F_j v_j\right) - \exp\left(d\sum_{j=0}^{s-1} F_j v_j\right). \end{aligned}$$

Assumption (13) implies that $\{\bar{T}_s\}$ is a Cauchy sequence in $\mathcal{B}(\mathfrak{X}_\infty)$ and so $T_\infty \in \mathcal{B}(\mathfrak{X}_\infty)$ exists.

To show (19) let us first verify the inequality

$$\|e^{\tilde{\Theta}_u^s}(K_u) - K_u\| \leq \frac{1+dB}{F_{\inf}}\left(e^{F_s w_s} - 1\right), \tag{21}$$

valid for all $u > s$. Actually, using definition (9) and assumption (11), we get

$$\begin{aligned}
\|e^{\tilde{\Theta}_u^s}(K_u) - K_u\| &\leq \frac{e^{\|\Theta_u^s\|} - 1}{\|\Theta_u^s\|}\|\tilde{\Theta}_u^s(K_u)\| \\
&\leq \frac{e^{\|\Theta_u^s\|} - 1}{\|\Theta_u^s\|}\left(\|\Theta_u^s\|\|W_s\| + \|W_s - \iota_{s-1}(W_{s-1})\|\right) \\
&\leq \left(e^{F_s w_s} - 1\right)\left(\|W_s\| + \frac{1}{F_s}\right).
\end{aligned}$$

To finish the estimate note that (13) and (16) imply

$$\|W_s\| = \sum_{j=1}^{s}(\|W_j\| - \|W_{j-1}\|) + \|W_0\| \leq \sum_{j=0}^{\infty} dv_j \leq \frac{d}{F_{\inf}}\sum_{j=0}^{\infty} F_j v_j = \frac{dB}{F_{\inf}}.$$

With the aid of an elementary identity,

$$a_j \dots a_0 - 1 = a_j \dots a_1(a_0 - 1) + a_j \dots a_2(a_1 - 1) + \dots + (a_j - 1),$$

we can derive from (21): if $0 \leq s \leq t < u$ then

$$\begin{aligned}
\|e^{\tilde{\Theta}_u^t} \dots e^{\tilde{\Theta}_u^s}(K_u) - K_u\| &\leq e^{\|\Theta_u^t\| + \dots + \|\Theta_u^{s+1}\|}\|e^{\tilde{\Theta}_u^s}(K_u) - K_u\| \\
&\quad + e^{\|\Theta_u^t\| + \dots + \|\Theta_u^{s+2}\|}\|e^{\tilde{\Theta}_u^{s+1}}(K_u) - K_u\| \\
&\quad + \dots + \|e^{\tilde{\Theta}_u^t}(K_u) - K_u\| \\
&\leq \frac{1+dB}{F_{\inf}}\Big(e^{F_t w_t + \dots + F_{s+1} w_{s+1}}\left(e^{F_s w_s} - 1\right) \\
&\quad + e^{F_t w_t + \dots + F_{s+2} w_{s+2}}\left(e^{F_{s+1} w_{s+1}} - 1\right) \\
&\quad + \dots + \left(e^{F_t w_t} - 1\right)\Big) \\
&= \frac{1+dB}{F_{\inf}}\left(e^{F_t w_t + \dots + F_s w_s} - 1\right).
\end{aligned}$$

Set temporarily in this proof

$$\tau_s = \iota_{\infty s}(\tilde{T}_s(K_s) - K_s) \in \mathfrak{X}_\infty.$$

If $t \geq s$ then

$$\begin{aligned}
\tau_t - \tau_s &= \iota_{\infty t}\left(e^{\tilde{\Theta}_t^{t-1}} \dots e^{\tilde{\Theta}_t^0}(K_t) - \iota_{ts} e^{\tilde{\Theta}_s^{s-1}} \dots e^{\tilde{\Theta}_s^0}(K_s)\right) \\
&= \iota_{\infty t}\left(e^{\tilde{\Theta}_t^{t-1}} \dots e^{\tilde{\Theta}_t^0}(K_t) - e^{\tilde{\Theta}_t^{s-1}} \dots e^{\tilde{\Theta}_t^0}(K_t)\right) \\
&= \iota_{\infty t}\Big(\left(e^{\Theta_t^{t-1}} \dots e^{\Theta_t^s} - 1\right)\left(e^{\tilde{\Theta}_t^{s-1}} \dots e^{\tilde{\Theta}_t^0}(K_t) - K_t\right) \\
&\quad + e^{\tilde{\Theta}_t^{t-1}} \dots e^{\tilde{\Theta}_t^s}(K_t) - K_t\Big).
\end{aligned}$$

Hence

$$\begin{aligned}\|\tau_t-\tau_s\| &\leq \frac{1+dB}{F_{\rm inf}}\Big(\big(e^{F_{t-1}w_{t-1}+\cdots+F_sw_s}-1\big)\big(e^{F_{s-1}w_{s-1}+\cdots+F_0w_0}-1\big)\\ &\quad+e^{F_{t-1}w_{t-1}+\cdots+F_sw_s}-1\Big)\\ &= \frac{1+dB}{F_{\rm inf}}\big(e^{F_{t-1}w_{t-1}+\cdots+F_0w_0}-e^{F_{s-1}w_{s-1}+\cdots+F_0w_0}\big).\end{aligned}$$

This shows that the sequence $\{\tau_s\}$ is Cauchy and thus the limit on the RHS of (19) exists.

We conclude that it holds true, in virtue of (10), that

$$\begin{aligned}\tilde{T}_\infty(K_\infty+V_\infty) &= K_\infty+\lim_{s\to\infty}\iota_{\infty s}(\tilde{T}_s(K_s)-K_s)+\lim_{s\to\infty}\bar{T}_s\iota_{\infty s}(V_s)\\ &= K_\infty+\lim_{s\to\infty}\iota_{\infty s}(\tilde{T}_s(K_s+V_s)-K_s)\\ &= K_\infty+\lim_{s\to\infty}\iota_{\infty s}\big(\mathcal{D}_s(W_s)+(1-\mathcal{D}_s)(W_s-\iota_{s-1}(W_{s-1}))\big)\\ &= K_\infty+\lim_{s\to\infty}\big(\mathcal{D}_\infty(\iota_{\infty s}(W_s))+(1-\mathcal{D}_\infty)(\iota_{\infty s}(W_s)-\iota_{\infty,s-1}(W_{s-1}))\big)\\ &= K_\infty+\mathcal{D}_\infty(W_\infty).\end{aligned}$$

So equality (20) has been verified as well. □

4. Convergence in a Hilbert space

Let $\{\mathfrak{X}_s,\iota_{us}\}$ be a directed sequence of real or complex Banach spaces, as introduced in Section 3. In this section it is sufficient to know that $\mathcal{K}$ is a separable complex Hilbert space and $\mathbf{K}$ is a closed (densely defined) operator in $\mathcal{K}$. Suppose that for each $s\in\mathbb{Z}_+$ there is given a bounded linear mapping,

$$\kappa_s:\mathfrak{X}_s\to\mathcal{B}(\mathcal{K}),\ \text{with}\ \|\kappa_s\|\leq 1,$$

and such that

$$\forall s,u,\ 0\leq s\leq u,\quad \kappa_u\iota_{us}=\kappa_s.$$

If the Banach spaces $\mathfrak{X}_s$ are real then the mappings κ_s are supposed to be linear over $\mathbb{R}$ otherwise they are linear over $\mathbb{C}$. Then there exists a unique linear bounded mapping $\kappa_\infty:\mathfrak{X}_\infty\to\mathcal{B}(\mathcal{K})$ satisfying, $\forall s\in\mathbb{Z}_+$, $\kappa_\infty\iota_{\infty s}=\kappa_s$. Clearly, $\|\kappa_\infty\|\leq 1$. Extend the mappings κ_s to $\tilde{\kappa}_s:\tilde{\mathfrak{X}}_s=\mathbf{k}K_s+\mathfrak{X}_s\to\mathbb{C}\mathbf{K}+\mathcal{B}(\mathcal{K})$ by defining

$$\tilde{\kappa}_s(K_s)=\mathbf{K},\ \forall s\in\mathbb{Z}_+\cup\{\infty\}.$$

So $\tilde{\kappa}_s(K_s+X)=\mathbf{K}+\kappa_s(X)$, with $X\in\mathfrak{X}_s$, is a closed operator in $\mathcal{K}$ with $\mathrm{Dom}(\mathbf{K}+\kappa_s(X))=\mathrm{Dom}(\mathbf{K})$.

Suppose, in addition, that there exists $\mathbf{D}\in\mathcal{B}(\mathcal{B}(\mathcal{K}))$ such that

$$\forall s\in\mathbb{Z}_+,\quad \mathbf{D}\kappa_s=\kappa_s\mathcal{D}_s.$$

Then it holds true, $\forall s\in\mathbb{Z}_+$, $\forall X\in\mathfrak{X}_s$,

$$\kappa_\infty\mathcal{D}_\infty(\iota_{\infty s}X)=\kappa_\infty\iota_{\infty s}\mathcal{D}_s(X)=\kappa_s\mathcal{D}_s(X)=\mathbf{D}\kappa_s(X)=\mathbf{D}\kappa_\infty(\iota_{\infty s}X).$$

Since the set of vectors $\{\iota_{\infty s}(X);\ s\in\mathbb{Z}_+,\ X\in\mathfrak{X}_s\}$ is dense in $\mathfrak{X}_s$ we get $\kappa_\infty\mathcal{D}_\infty=\mathbf{D}\kappa_\infty$.

Proposition 5. *Under the assumptions of Corollary 4 and those introduced above in this section, let* $\{\mathbf{A}_s\}_{s=0}^{\infty}$ *be a sequence of bounded operators in* $\mathcal{K}$ *such that,*

$$\forall s,u,\ 0\le s<u,\ \forall X\in\mathfrak{X}_u,\quad \kappa_u\big(\Theta_u^s(X)\big)=[\,\mathbf{A}_s,\kappa_u(X)\,],\tag{22}$$

$$\forall s\in\mathbb{Z}_+,\quad \mathbf{A}_s(\operatorname{Dom}\mathbf{K})\subset\operatorname{Dom}\mathbf{K},$$

and

$$\forall s,u,\ 0\le s<u,\quad [\,\mathbf{A}_s,\mathbf{K}\,]=\kappa_u(\tilde{\Theta}_u^s(K_u))\big|_{\operatorname{Dom}(\mathbf{K})}.$$

Moreover, assume that

$$\sum_{s=0}^{\infty}\|\mathbf{A}_s\|<\infty.\tag{23}$$

Set

$$\mathbf{V}=\kappa_\infty(V_\infty),\ \ \mathbf{W}=\kappa_\infty(W_\infty).$$

Then the limit

$$\mathbf{U}=\lim_{s\to\infty}e^{\mathbf{A}_{s-1}}\dots e^{\mathbf{A}_0}\tag{24}$$

exists in the operator norm, the element $\mathbf{U}\in\mathcal{B}(\mathcal{K})$ *has a bounded inverse, and it holds true that*

$$\mathbf{U}(\operatorname{Dom}\mathbf{K})=\operatorname{Dom}\mathbf{K}$$

and

$$\mathbf{U}(\mathbf{K}+\mathbf{V})\mathbf{U}^{-1}=\mathbf{K}+\mathbf{D}(\mathbf{W}).\tag{25}$$

For the proof we shall need a lemma.

Lemma 6. *Assume that* $\mathcal{H}$ *is a Hilbert space,* K *is a closed operator in* $\mathcal{H}$, $A,B\in\mathcal{B}(\mathcal{H})$,

$$A(\operatorname{Dom}K)\subset\operatorname{Dom}K,$$

and

$$[\,A,K\,]=B\big|_{\operatorname{Dom}(K)}.$$

Then it holds, $\forall\lambda\in\mathbb{C}$,

$$e^{\lambda A}(\operatorname{Dom}K)=\operatorname{Dom}K\tag{26}$$

and

$$e^{-\lambda A}Ke^{\lambda A}=K+\frac{e^{-\lambda\,\mathrm{ad}_A}-1}{\mathrm{ad}_A}\,B.$$

Remark. Here and everywhere in what follows we use the standard notation: $\mathrm{ad}_A B=[\,A,B\,]$ and so $e^{\lambda\,\mathrm{ad}_A}B=e^{\lambda A}B\,e^{-\lambda A}$.

Proof. Choose an arbitrary vector $v \in \mathrm{Dom}(K)$ and set

$$\forall n \in \mathbb{Z}_+, \quad v_n = \sum_{k=0}^{n} \frac{\lambda^k}{k!} A^k v\,.$$

Then $v_n \in \mathrm{Dom}(K)$ and $v_n \to e^{\lambda A} v$ as $n \to \infty$. On the other hand,

$$\begin{aligned} K v_n &= \sum_{k=0}^{n} \frac{\lambda^k}{k!} (K\,A^k - A^k K) v + \sum_{k=0}^{n} \frac{\lambda^k}{k!} A^k K v \\ &= -\sum_{k=1}^{n} \frac{\lambda^k}{k!} \sum_{j=0}^{k-1} A^j B\, A^{k-1-j} v + \sum_{k=0}^{n} \frac{\lambda^k}{k!} A^k K v\,. \end{aligned}$$

So the limit $\lim_{n\to\infty} K v_n$ exists. Consequently, since K is closed, $e^{\lambda A}(\mathrm{Dom}\,K) \subset \mathrm{Dom}\,K$. But $(e^{\lambda A})^{-1} = e^{-\lambda A}$ has the same property and thus equality (26) follows. Furthermore, the above computation also shows that

$$K\,e^{\lambda A} = -\sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \sum_{j=0}^{k-1} A^j B\, A^{k-1-j} + e^{\lambda A} K\,.$$

Application of the following algebraic identity (easy to verify),

$$\sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \sum_{j=0}^{k-1} A^j B\, A^{k-1-j} = e^{\lambda A} \left(\frac{1 - e^{-\lambda\, \mathrm{ad}_A}}{\mathrm{ad}_A} B \right),$$

concludes the proof. □

Proof of Proposition 5. We use notation of Corollary 4. From (22) follows that, $\forall s, u,\ 0 \le s < u$, $\forall X \in \mathfrak{X}_u$,

$$\kappa_\infty \Theta^s_\infty(\iota_{\infty u} X) = \kappa_u \Theta^s_u(X) = [\,\mathbf{A}_s, \kappa_u(X)\,] = [\,\mathbf{A}_s, \kappa_\infty(\iota_{\infty u} X)\,].$$

Since the set of vectors $\{\iota_{\infty u}(X);\ s < u,\ X \in \mathfrak{X}_u\}$ is dense in $\mathfrak{X}_\infty$, we get, $\forall X \in \mathfrak{X}_\infty$, $\kappa_\infty \Theta^s_\infty(X) = [\,\mathbf{A}_s, \kappa_\infty(X)\,]$, and hence

$$\kappa_\infty \left(e^{\Theta^s_\infty}(X)\right) = e^{\mathbf{A}_s} \kappa_\infty(X)\, e^{-\mathbf{A}_s}.$$

Set

$$\mathbf{U}_s = e^{\mathbf{A}_{s-1}} \ldots e^{\mathbf{A}_0} \text{ for } s \ge 1,\ \mathbf{U}_0 = 1.$$

Assumption (23) implies that both sequences $\{\mathbf{U}_s\}$ and $\{\mathbf{U}_s^{-1}\}$ are Cauchy in $\mathcal{B}(\mathcal{K})$ and hence the limit (24) exists in the operator norm, with $\mathbf{U}^{-1} = \lim_{s\to\infty} \mathbf{U}_s^{-1} \in \mathcal{B}(\mathcal{K})$. Moreover, $\forall X \in \mathfrak{X}_\infty$,

$$\kappa_\infty T_\infty(X) = \kappa_\infty \left(\lim_{s\to\infty} e^{\Theta^{s-1}_\infty} \ldots e^{\Theta^0_\infty} X \right) = \lim_{s\to\infty} \mathbf{U}_s \kappa_\infty(X)\, \mathbf{U}_s^{-1}. \tag{27}$$

Next let us compute $\tilde{\kappa}_s \tilde{T}_s(K_s)$. For $0 \le s < u$, set $\mathbf{B}_s = \kappa_u(\tilde{\Theta}^s_u(K_u)) \in \mathcal{B}(\mathcal{K})$. $\mathbf{B}_s$ doesn't depend on $u > s$ since if $0 \le s < u \le v$ then

$$\kappa_u\left(\tilde{\Theta}^s_u(K_u)\right) = \kappa_v\left(\iota_{vu} \tilde{\Theta}^s_u(K_u)\right) = \kappa_v\left(\tilde{\Theta}^s_v(K_v)\right).$$

We can apply Lemma 6 to the operators $\mathbf{K}$, $\mathbf{A}_s$, $\mathbf{B}_s$ to conclude that $e^{-\mathbf{A}_s}(\mathrm{Dom}\,\mathbf{K}) = \mathrm{Dom}\,\mathbf{K}$ and

$$e^{\mathbf{A}_s}\mathbf{K}\,e^{-\mathbf{A}_s} = \mathbf{K} + \frac{e^{\mathrm{ad}_{\mathbf{A}_s}} - 1}{\mathrm{ad}_{\mathbf{A}_s}}\,\mathbf{B}_s. \tag{28}$$

On the other hand,

$$\tilde{\kappa}_u\left(e^{\tilde{\Theta}^s_u}(K_u)\right) = \tilde{\kappa}_u\left(K_u + \frac{e^{\Theta^s_u} - 1}{\Theta^s_u}\tilde{\Theta}^s_u(K_u)\right) = \mathbf{K} + \frac{e^{\mathrm{ad}_{\mathbf{A}_s}} - 1}{\mathrm{ad}_{\mathbf{A}_s}}\,\mathbf{B}_s.$$

Thus $\tilde{\kappa}_u\left(e^{\tilde{\Theta}^s_u}(K_u)\right) = e^{\mathbf{A}_s}\mathbf{K}\,e^{-\mathbf{A}_s}$. Consequently, $\mathbf{U}_s(\mathrm{Dom}\,\mathbf{K}) = \mathrm{Dom}\,\mathbf{K}$ and

$$\tilde{\kappa}_s\tilde{T}_s(K_s) = \mathbf{U}_s\mathbf{K}\,\mathbf{U}_s^{-1}. \tag{29}$$

Set $\mathbf{C}_s = \mathbf{U}_s\mathbf{K}\mathbf{U}_s^{-1} - \mathbf{K}$. According to (28), $\mathbf{C}_s \in \mathcal{B}(\mathcal{K})$. Now we can compute, using relation (29), a limit in $\mathcal{B}(\mathcal{K})$,

$$\begin{aligned}
\mathbf{C} &= \lim_{s\to\infty}\mathbf{C}_s = \lim_{s\to\infty}\kappa_s(\tilde{T}_s(K_s) - K_s)\\
&= \kappa_\infty\left(\lim_{s\to\infty}\iota_{\infty s}(\tilde{T}_s(K_s) - K_s)\right)\\
&= \kappa_\infty(\tilde{T}_\infty(K_\infty) - K_\infty).
\end{aligned}$$

So $\mathbf{K} + \mathbf{C} = \tilde{\kappa}_\infty(\tilde{T}_\infty(K_\infty))$. From the closeness of $\mathbf{K}$, the equality $\mathbf{U}_s\mathbf{K}\mathbf{U}_s^{-1} = \mathbf{K} + \mathbf{C}_s$, and from the fact that the sequences $\{\mathbf{U}_s^{\pm 1}\}$, $\{\mathbf{C}_s\}$ converge one deduces that $\mathbf{U}^{\pm 1}(\mathrm{Dom}\,\mathbf{K}) \subset \mathrm{Dom}\,\mathbf{K}$ and hence, in fact, $\mathbf{U}^{\pm 1}(\mathrm{Dom}\,\mathbf{K}) = \mathrm{Dom}\,\mathbf{K}$. In addition,

$$\mathbf{U}\mathbf{K}\mathbf{U}^{-1} = \mathbf{K} + \mathbf{C} = \tilde{\kappa}_\infty\tilde{T}_\infty(K_\infty). \tag{30}$$

Combining (27) and (30) one finds that

$$\tilde{\kappa}_\infty\tilde{T}_\infty(X) = \mathbf{U}\tilde{\kappa}_\infty(X)\mathbf{U}^{-1},\ \forall X \in \tilde{\mathfrak{X}}_\infty.$$

To conclude the proof it suffices to apply the mapping $\tilde{\kappa}_\infty$ to equality (20). □

5. Choice of the directed sequence of Banach spaces

Suppose that there are given a decreasing sequence of subsets of the interval $]0,+\infty[$, $\Omega_0 \supset \Omega_1 \supset \Omega_2 \supset \ldots$, a decreasing sequence of positive real numbers $\{\varphi_s\}_{s=0}^\infty$ and a strictly increasing sequence of positive real numbers $\{E_s\}_{s=0}^\infty$, $1 \le E_1 < E_2 < \ldots$.
We construct a complex Banach space ${}^0\mathfrak{X}_s$, $s \ge 0$, as a subspace

$${}^0\mathfrak{X}_s \subset L^\infty\left(\Omega_s \times \mathbb{Z} \times \mathbb{N} \times \mathbb{N},\ \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}}^{\oplus}\mathcal{B}(\mathcal{H}_m, \mathcal{H}_n)\right)$$

formed by those elements $X = \{X_{knm}(\omega)\}$ which satisfy

$$X_{knm}(\omega) \in \mathcal{B}(\mathcal{H}_m, \mathcal{H}_n),\ \forall \omega \in \Omega_s,\ \forall (k,n,m) \in \mathbb{Z}\times\mathbb{N}\times\mathbb{N},$$

and have finite norm

$$\|X\|_s = \sup_{\substack{\omega,\omega'\in\Omega_s\\ \omega\neq\omega'}}\ \sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{N}}\left(\|X_{knm}(\omega)\| + \varphi_s\,\|\partial X_{knm}(\omega,\omega')\|\right)e^{|k|/E_s} \tag{31}$$

where the symbol ∂ designates the discrete derivative in ω,

$$\partial X(\omega,\omega') = \frac{X(\omega)-X(\omega')}{\omega-\omega'}\,.$$

In fact, this norm is considered in Appendix B (c.f. (87)), and it is shown there that ${}^0\mathfrak{X}_s$ is an operator algebra with respect to the multiplication rule (89).
Let $\mathfrak{X}_s \subset {}^0\mathfrak{X}_s$ be a closed real subspace formed by those elements $X \in {}^0\mathfrak{X}_s$ which satisfy,

$$\forall (k,n,m) \in \mathbb{Z}\times\mathbb{N}\times\mathbb{N},\ \forall\omega\in\Omega_s,\quad X_{knm}(\omega)^* = X_{-k,m,n}(\omega) \in \mathcal{B}(\mathcal{H}_n,\mathcal{H}_m). \tag{32}$$

Note, however, that $\mathfrak{X}_s$ is not an operator subalgebra of ${}^0\mathfrak{X}_s$.
The sequence of Banach spaces, $\{\mathfrak{X}_s\}_{s=0}^{\infty}$, becomes directed with respect to mappings of restriction in the variable ω: if $u \geq s$ then we set

$$\iota_{us} : \mathfrak{X}_s \to \mathfrak{X}_u,\ \ \iota_{us}(X) = X|_{\Omega_u}.$$

Because of the monotonicity of the sequences $\{\varphi_s\}$ and $\{E_s\}$ we clearly have $\|\iota_{us}\| \leq 1$.
Next we introduce a bounded operator $\mathcal{D}_s \in \mathcal{B}(\mathfrak{X}_s)$ as an operator which extracts the diagonal part of a matrix,

$$\mathcal{D}_s(X)_{knm}(\omega) = \delta_{k0}\delta_{nm}X_{0nn}(\omega). \tag{33}$$

Clearly, $\|\mathcal{D}_s\| \leq 1$ and $\|1-\mathcal{D}_s\| \leq 1$.
Let

$$V \in L^\infty\left(\mathbb{Z}\times\mathbb{N}\times\mathbb{N},\ \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}}{}^{\oplus}\,\mathcal{B}(\mathcal{H}_m,\mathcal{H}_n)\right)$$

be the element with the components $V_{knm} \in \mathcal{B}(\mathcal{H}_m,\mathcal{H}_n)$ given in (2). Since, by assumption, $V(t)$ is Hermitian for almost all t it hold true that

$$(V_{knm})^* = V_{-k,m,n}.$$

We still assume, as in Theorem 1, that there exists $r > 0$ such that

$$\epsilon_V = \sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{N}}\|V_{knm}\|\ \max\{|k|^r,1\} < \infty\,. \tag{34}$$

Let us define elements $V_s \in \mathfrak{X}_s$, $s \geq 0$, by

$$\begin{aligned}(V_s)_{knm}(\omega) &= V_{knm} &&\text{if } |k| < E_s\\ &= 0 &&\text{if } |k| \geq E_s\end{aligned} \tag{35}$$

For $s \geq 1$ we get an estimate,

$$\begin{aligned}\|V_s - \iota_{s-1}(V_{s-1})\|_s &= \sup_{n\in\mathbb{N}} \sum_{\substack{k\in\mathbb{Z}\\ E_{s-1}\leq|k|<E_s}} \sum_{m\in\mathbb{N}} \|V_{knm}\|\, e^{|k|/E_s}\\ &\leq e\,\sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{N}}\|V_{knm}\|\,\frac{\max\{|k|^r,1\}}{(E_{s-1})^r}\\ &= \frac{e\,\epsilon_V}{(E_{s-1})^r}\,.\end{aligned} \tag{36}$$

Similarly, for $s = 0$, we get

$$\|V_0\| \leq e\,\epsilon_V.$$

It is convenient to set $E_{-1}=1$, $V_{-1}=0$.
The sequence $\{K_s\}_{s=0}^{\infty}$ has the same meaning as in Section 3, i.e., each K_s is a distinguished basis vector in a one-dimensional vector space $\mathbb{R}K_s$. Furthermore, a sequence $\Theta_u^s \in \mathcal{B}(\mathfrak{X}_u)$, $0 \le s < u$, is supposed to satisfy rule (6). Similarly as in Proposition 2 we construct sequences $T_s \in \mathcal{B}(\mathfrak{X}_s)$, $s \ge 1$, and $W_s \in \mathfrak{X}_s$, $s \ge 0$, using relations (7) and (8), respectively.

Proposition 7. *Suppose that it holds*

$$\|\Theta_u^s\| \le \frac{5}{\varphi_{s+1}}\,\|W_s - \iota_{s-1}(W_{s-1})\|_s,\ \forall s,u,\ 0 \le s < u, \tag{37}$$

and set

$$A_\star = 5e\sup_{s\ge 0}\frac{(E_s)^r}{\varphi_{s+1}(E_{s-1})^{2r}},\ B_\star = 5e\sum_{s=0}^{\infty}\frac{1}{\varphi_{s+1}(E_{s-1})^r},\ C_\star = 5e\sup_{s\ge 0}\frac{1}{\varphi_{s+1}(E_{s-1})^r}\,. \tag{38}$$

If

$$\epsilon_V B_\star \le \frac{1}{3}\ln 2 \quad \textit{and} \quad \epsilon_V A_\star \phi(3\epsilon_V C_\star) \le \frac{1}{9} \tag{39}$$

then the conclusions of Corollary 3 hold true, particularly, the objects $V_\infty, W_\infty \in \mathfrak{X}_\infty$, $T_\infty \in \mathcal{B}(\mathfrak{X}_\infty)$ and $\tilde{T}_\infty \in \mathcal{B}(\tilde{\mathfrak{X}}_\infty)$ exist and satisfy the equality

$$\tilde{T}_\infty(K_\infty + V_\infty) = K_\infty + \mathcal{D}_\infty(W_\infty)\,.$$

Remark. Respecting estimates (36) and (37) we set in what follows

$$F_s = \frac{5}{\varphi_{s+1}} \text{ and } v_s = \frac{e\,\epsilon_V}{(E_{s-1})^r},\ s \ge 0. \tag{40}$$

Proof. Taking into account the defining relations (40) one finds that the constants A, B and C introduced in Proposition 3 may be chosen as

$$A = \epsilon_V A_\star,\ B = \epsilon_V B_\star \text{ and } C = \epsilon_V C_\star. \tag{41}$$

The assumption (39) implies that

$$B \le \frac{1}{3}\ln 2 \quad \text{and} \quad A\,\phi(3C) \le \frac{1}{9} \tag{42}$$

and so, according to the remark following Proposition 3, inequality (15) holds true with $d=3$. Since $F_{\inf} = 5/\varphi_1 > 0$ assumption (18) of Corollary 4 as well as all assumptions of Proposition 3 are satisfied and so the conclusions of Corollary 4 hold true. □

6. Relation of the Banach spaces $\mathfrak{X}_s$ to Hermitian operators in $\mathcal{K}$

The real Banach spaces $\mathfrak{X}_s$ have been chosen in the previous section. Set

$$\Omega_\infty = \bigcap_{s=0}^{\infty}\Omega_s.$$

Suppose that $\Omega_\infty \ne \emptyset$ and fix $\omega \in \Omega_\infty$ (so $\omega > 0$).
To an operator-valued function $[\,0,T\,] \ni t \mapsto X(t) \in \mathcal{B}(\mathcal{H})$ there is naturally related an operator $\mathbf{X}$ in $\mathcal{K} = L^2([\,0,T\,],\mathcal{H},\mathrm{d}t)$ defined by $(\mathbf{X}\psi)(t) = X(t)\psi(t)$. As is well known,

$$\|\mathbf{X}\| \le \|X\|_{SH}$$

where $\|\cdot\|_{SH}$ is the so called Schur-Holmgren norm,

$$\begin{aligned}\|X\|_{SH} &= \max\Bigg\{\sup_{(\ell,n)\in\mathbb{Z}\times\mathbb{N}}\sum_{(k,m)\in\mathbb{Z}\times\mathbb{N}}\big\|P_\ell\otimes Q_n\mathbf{X}P_k\otimes Q_m\big\|\,,\\ &\qquad \sup_{(k,m)\in\mathbb{Z}\times\mathbb{N}}\sum_{(\ell,n)\in\mathbb{Z}\times\mathbb{N}}\big\|P_\ell\otimes Q_n\mathbf{X}P_k\otimes Q_m\big\|\Bigg\} \qquad (43)\\ &= \max\left\{\sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{N}}\|X_{knm}\|,\ \sup_{m\in\mathbb{N}}\sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{N}}\|X_{knm}\|\right\}.\end{aligned}$$

Here

$$X_{knm}=\frac{1}{T}\int_0^T e^{-i\omega kt}\,Q_nX(t)Q_m\,\mathrm{d}t\,.$$

It is also elementary to verify that the Schur-Holmgren norm is an operator norm, $\|XY\|_{SH}\le\|X\|_{SH}\|Y\|_{SH}$, with respect to the multiplication rule (89).
If $X(t)$ is Hermitian for (almost) every $t\in[\,0,T\,]$ then it holds, $\forall(k,n,m)$, $(X_{knm})^*=X_{-k,m,n}$, and so

$$\|X\|_{SH}=\sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{N}}\|X_{knm}\|\,.$$

Note also that, $\forall s\in\mathbb{Z}_+$, $\forall X\in\mathfrak{X}_s$,

$$\|X(\omega)\|_{SH}\le\|X\|_s$$

and, consequently, the same is also true for $s=\infty$.
To an element $X\in{}^0\mathfrak{X}_s\subset L^\infty\left(\Omega_s\times\mathbb{Z}\times\mathbb{N}\times\mathbb{N},\ \sum_{n\in\mathbb{N}}\sum^{\oplus}_{m\in\mathbb{N}}\mathcal{B}(\mathcal{H}_m,\mathcal{H}_n)\right)$ such that $\|X(\omega)\|_{SH}<\infty$ we can relate an operator-valued function defined on the interval $[\,0,T\,]$,

$$t\mapsto\sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}}e^{ik\omega t}\,X_{knm}(\omega)\,.$$

The corresponding operator in $\mathcal{K}$ is denoted by $\kappa_s(X)$, with a norm being bounded from above by $\|X(\omega)\|_{SH}$. In particular, $\forall X\in\mathfrak{X}_s$,

$$\|\kappa_s(X)\|\le\|X(\omega)\|_{SH}\le\|X\|_s.$$

In addition, if $X\in\mathfrak{X}_s$ then the operator $\kappa_s(X)$ is Hermitian due to the property (32) of X. This way we have introduced the mappings $\kappa_s:\mathfrak{X}_s\to\mathcal{B}(\mathcal{K})$ for $s\in\mathbb{Z}_+$.
Another property we shall need is that κ_s is an algebra morphism in the sense: if $X,Y\in{}^0\mathfrak{X}_s$ such that $\|X(\omega)\|_{SH}<\infty$ and $\|Y(\omega)\|_{SH}<\infty$ then $\|(XY)(\omega)\|_{SH}<\infty$ and

$$\kappa_s(XY)=\kappa_s(X)\kappa_s(Y)\,.$$

Particularly this is true for all $X,Y\in\mathfrak{X}_s$.

Let $\mathbf{D} \in \mathcal{B}(\mathcal{B}(\mathcal{K}))$ be the operator on $\mathcal{B}(\mathcal{K})$ taking the diagonal part of an operator $X \in \mathcal{B}(\mathcal{K})$,

$$\mathbf{D}(X) = \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{N}} P_k \otimes Q_m X\, P_k \otimes Q_m .$$

Clearly, $\mathbf{D}\kappa_s = \kappa_s \mathcal{D}_s$. Since

$$\|\mathbf{D}(X)\| = \sup_{(k,m)\in\mathbb{Z}\times\mathbb{N}} \|P_k \otimes Q_m X\, P_k \otimes Q_m\| \leq \|X\|$$

we have $\|\mathbf{D}\| \leq 1$.
A consequence of (34) is that $V = \{V_{knm}\}$ has a a finite Schur-Holmgren norm, $\|V\|_{SH} < \infty$. Let $V_s \in \mathfrak{X}_s$, $s \in \mathbb{Z}_+$, be the cut-offs of V defined in (35). Then

$$\begin{aligned}
\|V - V_s\|_{SH} &= \sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{Z},|k|\geq E_s} \sum_{m\in\mathbb{N}} \|V_{knm}\| \\
&\leq \frac{1}{(E_s)^r} \sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{N}} \|V_{knm}\| \max\{|k|^r, 1\} \\
&= \frac{\epsilon_V}{(E_s)^r} .
\end{aligned}$$

We shall impose an additional condition on the increasing sequence $\{E_s\}$ of positive real numbers that occur in the definition of the norm $\|\cdot\|_s$ in $\mathfrak{X}_s$ (c.f. (31)), namely we shall require

$$\lim_{s\to\infty} E_s = +\infty . \tag{44}$$

In this case $\lim_{s\to\infty} \|V - V_s\|_{SH} = 0$ and so

$$\mathbf{V} = \lim_{s\to\infty} \kappa_s(V_s) \quad \text{in the operator norm.} \tag{45}$$

We also assume that there exist $A_s \in \mathfrak{X}_{s+1}$, $s \in \mathbb{Z}_+$, such that

$$(A_s)_{knm}(\omega)^* = -(A_s)_{-k,m,n}(\omega), \tag{46}$$

and, using these elements, we define mappings ${}^0\Theta_u^s \in \mathcal{B}({}^0\mathfrak{X}_u)$, $u > s$, by

$${}^0\Theta_u^s(X) = [\, \iota_{u,s+1}(A_s), X\,] \tag{47}$$

(where the commutator on the RHS makes sense since ${}^0\mathfrak{X}_u$ is an operator algebra). Clearly, $\|{}^0\Theta_u^s\| \leq 2\|A_s\|_{s+1}$. One finds readily that $\mathfrak{X}_u \subset {}^0\mathfrak{X}_u$ is an invariant subspace with respect to the mapping ${}^0\Theta_u^s$ and so one may define $\Theta_u^s = {}^0\Theta_u^s\big|_{\mathfrak{X}_u} \in \mathcal{B}(\mathfrak{X}_u)$. Since $iA_s \in \mathfrak{X}_{s+1}$ we can set

$$\mathbf{A}_s = -i\, \kappa_{s+1}(iA_{s+1}) \in \mathcal{B}(\mathcal{K}).$$

Clearly, $\mathbf{A}_s$ is anti-Hermitian and satisfies $\|\mathbf{A}_s\| \leq \|A_s\|_{s+1}$. Note that (47) implies that, $\forall s, u,\ 0 \leq s < u$, $\forall X \in \mathfrak{X}_u$,

$$\kappa_u\big(\Theta_u^s(X)\big) = [\, \mathbf{A}_s, \kappa_u(X)\,] .$$

Lemma 8. *Let $\{W_s\}_{s=0}^{\infty}$ be a sequence of elements $W_s \in \mathfrak{X}_s$ and let $\tilde{\Theta}_u^s : \tilde{\mathfrak{X}}_u \to \tilde{\mathfrak{X}}_u$ be the extension of Θ_u^s, $0 \leq s < u$, defined in (9). Assume that the elements $A_s \in {}^0\mathfrak{X}_{s+1}$, $s \in \mathbb{Z}_+$, satisfy*

$$\begin{aligned} &(k\omega - \Delta_{mn})\,(A_s)_{knm}(\omega) \\ &\quad = \big(\Theta_u^s\big(\iota_{us}\mathcal{D}_s(W_s)\big) + \iota_{us}(1-\mathcal{D}_s)\big(W_s - \iota_{s-1}(W_{s-1})\big)\big)_{knm}(\omega), \end{aligned} \tag{48}$$

$\forall (k,m,n) \in \mathbb{Z}\times\mathbb{N}\times\mathbb{N}$, $\forall s,u,\ 0 \leq s < u$.
Then it holds true that,

$$\forall s \in \mathbb{Z}_+, \quad \mathbf{A}_s(\mathrm{Dom}\,\mathbf{K}) \subset \mathrm{Dom}\,\mathbf{K},$$

and

$$\forall s,u,\ 0 \leq s < u, \quad [\,\mathbf{A}_s, \mathbf{K}\,] = \kappa_u(\tilde{\Theta}_u^s(K_u))\big|_{\mathrm{Dom}(\mathbf{K})}.$$

Proof. Set

$$\mathbf{B}_s = -\kappa_u\big(\tilde{\Theta}_u^s(K_u)\big).$$

Since the RHS of (48) is in fact a matrix entry of $-\tilde{\Theta}_u^s(K_u)$ (c.f. (9)) this assumption may be rewritten as the equality

$$\mathbf{K}\,P_\ell\otimes Q_n\mathbf{A}_sP_k\otimes Q_m = P_\ell\otimes Q_n\mathbf{A}_sP_k\otimes Q_m\mathbf{K} + P_\ell\otimes Q_n\mathbf{B}_sP_k\otimes Q_m,$$

valid for all $(\ell,n),(k,m) \in \mathbb{Z}\times\mathbb{N}$. Since $\mathbf{K}$ is closed one easily derives from the last property that it holds true, $\forall (k,m) \in \mathbb{Z}\times\mathbb{N}$,

$$\mathbf{K}\,\mathbf{A}_sP_k\otimes Q_m = \mathbf{A}_sP_k\otimes Q_m\mathbf{K} + \mathbf{B}_sP_k\otimes Q_m. \tag{49}$$

Particularly, $\mathbf{A}_s\,\mathrm{Ran}(P_k\otimes Q_m) \subset \mathrm{Dom}(\mathbf{K})$. But $\mathrm{Ran}(P_k\otimes Q_m)$ are mutually orthogonal eigenspaces of $\mathbf{K}$. Consequently, if $v \in \mathrm{Dom}(\mathbf{K})$, then the sequence $\{v_N\}_{N=1}^{\infty}$,

$$v_N = \sum_{k,\,|k|\leq N}\ \sum_{m,\,m\leq N} P_k\otimes Q_m v$$

has the property: $v_N \to v$ and $\mathbf{K}v_N \to \mathbf{K}v$, as $N \to \infty$. Equality (49) implies that

$$\mathbf{K}\mathbf{A}_sv_N = \mathbf{A}_s\mathbf{K}v_N + \mathbf{B}_sv_N,\ \forall N.$$

Again owing to the fact that $\mathbf{K}$ is closed one concludes that $\mathbf{A}_sv \in \mathrm{Dom}(\mathbf{K})$ and $\mathbf{K}\mathbf{A}_sv = \mathbf{A}_s\mathbf{K}v + \mathbf{B}_sv$. □

Proposition 9. *Assume that $\omega \in \Omega_\infty$ and the norms $\|\cdot\|_s$ in the Banach spaces $\mathfrak{X}_s$ satisfy (44). Let $\Theta_u^s \in \mathcal{B}(\mathbf{X}_u)$, $0 \leq s < u$, be the operators defined in (47) with the aid of elements $A_s \in {}^0\mathfrak{X}_{s+1}$ satisfying (46), and let $W_s \in \mathfrak{X}_s$, $s \in \mathbb{Z}_+$, be a sequence defined recursively in accordance with (8). Assume that the elements A_s, $s \in \mathbb{Z}_+$, satisfy condition (48) and that*

$$\|A_s\| \leq \frac{5}{2\varphi_{s+1}}\,\|W_s - \iota_{s-1}(W_{s-1})\|,\ \forall s \in \mathbb{Z}_+. \tag{50}$$

Moreover, assume that the numbers $A_\star$, $B_\star$, $C_\star$, as defined in (38), satisfy condition (39).
Then there exist, in $\mathcal{K}$, a unitary operator $\mathbf{U}$ and a bounded Hermitian operator $\mathbf{W}$ such that

$$\mathbf{U}(\mathrm{Dom}\,\mathbf{K}) = \mathrm{Dom}\,\mathbf{K}$$

and

$$\mathbf{U}(\mathbf{K}+\mathbf{V})\mathbf{U}^{-1}=\mathbf{K}+\mathbf{D}(\mathbf{W}).$$

Proof. The norm of Θ_u^s may be estimated as

$$\|\Theta_u^s\|\leq 2\|A_s\|\leq\frac{5}{\varphi_{s+1}}\,\|W_s-\iota_{s-1}(W_{s-1})\|.$$

This way the assumptions of Proposition 7 are satisfied and consequently, according to Proposition 7 (and its proof), the same is true for Proposition 3 and Corollary 4 (with F_s and v_s defined in (40) and the constants A, B, C defined in (41)). Since it holds $\|\mathbf{A}_s\|\leq\|A_s\|\leq\frac{1}{2}F_sw_s$ (where $F_s=5/\varphi_{s+1}$) and, by assumption, condition (15) is satisfied with $d=3$ we get

$$\sum_{s=0}^{\infty}\|\mathbf{A}_s\|\leq\frac{1}{2}\sum_{s=0}^{\infty}F_sw_s\leq\frac{3}{2}\sum_{s=0}^{\infty}F_sv_s=\frac{3B}{2}<\infty.$$

This verifies assumption (23) of Proposition 5; the other assumptions of this proposition are verified as well as follows from Lemma 8. Note that, in virtue of (45), $\kappa_\infty(V_\infty)=\lim_{s\to\infty}\kappa_s(V_s)$ coincides with the given operator $\mathbf{V}$. Furthermore, $\mathbf{W}=\kappa_\infty(W_\infty)=\lim_{s\to\infty}\kappa_s(W_s)$ is a limit of Hermitian operators and so is itself Hermitian, and $\mathbf{U}=\lim_{s\to\infty}e^{\mathbf{A}_{s-1}}\dots e^{\mathbf{A}_0}$ is unitary. Equality (25) holds true and this concludes the proof. □

7. Set of non-resonant frequencies

Let $J>0$ be fixed and assume that, $\forall s\in\mathbb{Z}_+$,

$$\Omega_s\subset\Big[\frac{8}{9}J,\frac{9}{8}J\Big].$$

The following definition concerns indices (k,n,m) corresponding to non-diagonal entries, i.e., those indices for which either $k\neq 0$ or $m\neq n$. The diagonal indices, with $k=0$ and $m=n$, will always be treated separately and, in fact, in a quite trivial manner.

Definition. We shall say that a multi-index $(k,n,m)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}$ is *critical* if $m\neq n$ and

$$\frac{kJ}{\Delta_{mn}}\in\Big]\frac{1}{2},2\Big[\tag{51}$$

(hence $\operatorname{sgn}(k)=\operatorname{sgn}(h_m-h_n)\neq 0$). In the opposite case the multi-index will be called *non-critical.*

Definition. Let $\psi(k,n,m)$ be a positive function defined on non-diagonal indices and $W\in\mathfrak{X}_s$. A frequency $\omega\in\Omega_s$ will be called (W,ψ)*–non-resonant* if for all non-diagonal indices $(k,n,m)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}$ it holds

$$\operatorname{dist}\left(\operatorname{Spec}(k\omega-\Delta_{mn}+W_{0nn}(\omega)),\,\operatorname{Spec}(W_{0mm}(\omega))\right)\geq\psi(k,n,m). \tag{52}$$

In the opposite case ω will be called (W,ψ)*–resonant.*

Note that, in virtue of (32), $W_{0mm}(\omega)$ is a Hermitian operator in $\mathcal{H}_m$.

Lemma 10. *Assume that $\Omega_s \subset [\,\frac{8}{9}J, \frac{9}{8}J\,]$, $W \in \mathfrak{X}_s$ and ψ is a positive function defined on non-diagonal indices and obeying a symmetry condition,*

$$\psi(-k,m,n) = \psi(k,n,m) \quad \textit{for all } (k,n,m) \textit{ non-diagonal.} \tag{53}$$

If

$$\forall m \in \mathbb{N},\ \ \forall \omega,\omega' \in \Omega_s,\ \ \omega \neq \omega',\ \ \|\partial W_{0mm}(\omega,\omega')\| \leq \frac{1}{4}, \tag{54}$$

and if condition (52) is satisfied for all $\omega \in \Omega_s$ and all non-critical indices (k,n,m) then the Lebesgue measure of the set $\Omega_s^{\mathrm{bad}} \subset \Omega_s$ formed by (W,ψ)–resonant frequencies may be estimated as

$$|\Omega_s^{\mathrm{bad}}| \leq 8 \sum_{\substack{m,n\in\mathbb{N},\\ \Delta_{mn} > \frac{1}{2}J}} \ \sum_{\substack{k\in\mathbb{N},\\ \frac{\Delta_{mn}}{2J} < k < \frac{2\Delta_{mn}}{J}}} \frac{M_m M_n}{k}\, \psi(k,n,m)\,. \tag{55}$$

Proof. Let $\lambda_1^m(\omega) \leq \lambda_2^m(\omega) \leq \cdots \leq \lambda_{M_m}^m(\omega)$ be the increasingly ordered set of eigenvalues of $W_{0mm}(\omega)$, $m \in \mathbb{N}$. Set

$$\Omega_s^{\mathrm{bad}}(k,n,m,i,j) = \{\omega \in \Omega_s;\ |\omega k - \Delta_{mn} + \lambda_i^n(\omega) - \lambda_j^m(\omega)| < \psi(k,n,m)\}.$$

Then

$$\Omega_s^{\mathrm{bad}} = \bigcup_{(k,n,m)} \ \bigcup_{\substack{i,j\\ 1\leq i\leq M_n\\ 1\leq j\leq M_m}} \Omega_s^{\mathrm{bad}}(k,n,m,i,j).$$

By assumption, if (k,n,m) is a non-critical index then $\Omega_s^{\mathrm{bad}}(k,n,m,i,j) = \emptyset$ (for any i,j). Further notice that, due to the symmetry condition (53), $\Omega_s^{\mathrm{bad}}(k,n,m,i,j) = \Omega_s^{\mathrm{bad}}(-k,m,n,j,i)$.
According to Lidskii Theorem ([21], Chap. II §6.5), for any j, $1 \leq j \leq M_m$, $\lambda_j^m(\omega) - \lambda_j^m(\omega')$ may be written as a convex combination (with non-negative coefficients) of eigenvalues of the operator $W_{0mm}(\omega) - W_{0mm}(\omega')$. Consequently,

$$\forall j,\ 1 \leq j \leq M_m,\ \forall \omega,\omega' \in \Omega_s,\ \omega \neq \omega',\ |\partial\lambda_j^m(\omega,\omega')| \leq \|\partial W_{0mm}(\omega,\omega')\| \leq \frac{1}{4}\,.$$

If $\omega,\omega' \in \Omega_s^{\mathrm{bad}}(k,n,m,i,j)$, $\omega \neq \omega'$, then (k,n,m) is necessarily a critical index and

$$\begin{aligned}\frac{2\psi(k,n,m)}{|\omega-\omega'|} &> \left|\frac{(\omega k - \Delta_{mn} + \lambda_i^n(\omega) - \lambda_j^m(\omega)) - (\omega' k - \Delta_{mn} + \lambda_i^n(\omega') - \lambda_j^m(\omega'))}{\omega-\omega'}\right|\\ &\geq |k| - \frac{1}{2} \geq \frac{1}{2}\,|k|\,.\end{aligned}$$

This implies that $|\Omega_s^{\mathrm{bad}}(k,n,m,i,j)| \leq 4\psi(k,n,m)/|k|$ and so

$$|\Omega_s^{\mathrm{bad}}| \ \leq\ 2 \sum_{\substack{(k,n,m)\\ k>0\\ \frac{\Delta_{mn}}{2J}<k<\frac{2\Delta_{mn}}{J}}} \ \sum_{\substack{i,j\\ 1\leq i\leq M_n\\ 1\leq j\leq M_m}} \frac{4}{k}\,\psi(k,n,m)\,.$$

This immediately leads to the desired inequality (55). □

8. Construction of the sequences $\{\Omega_s\}$ and $\{A_s\}$

For a non-diagonal multi-index (k,n,m) and $s\in\mathbb{Z}_+$ set

$$\begin{aligned}\psi_s(k,n,m) &= \frac{1}{2}\,\Delta_0 && \text{if } (k,n,m) \text{ is non-critical and } k=0,\\ &= \frac{7}{18}\,J\left(|k|-\frac{1}{2}\right) && \text{if } (k,n,m) \text{ is non-critical and } k\neq 0,\\ &= \varphi_{s+1}\left(\min\{M_m,M_n\}\right)^{1/2}|k|^{1/2}e^{-\varrho_s|k|/2} && \text{if } (k,n,m) \text{ is critical,}\end{aligned} \tag{56}$$

where

$$\varrho_s=\frac{1}{E_s}-\frac{1}{E_{s+1}}.$$

Observe that ψ_s obeys the symmetry condition (53). The choice of $\psi_s(k,n,m)$ for a non-critical index (k,n,m) was guided by the following lemma.

Lemma 11. *If $\omega\in\Omega_s\subset[\,\frac{8}{9}J,\frac{9}{8}J\,]$, $(k,n,m)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}$ is a non-critical index and $W\in\mathfrak{X}_s$ satisfies*

$$\|W_{0mm}(\omega)\|,\|W_{0nn}(\omega)\|\leq\min\left\{\frac{1}{4}\,\Delta_0,\frac{7}{72}\,J\right\} \tag{57}$$

then the spectra $\mathrm{Spec}(k\omega-\Delta_{mn}+W_{0nn}(\omega))$, $\mathrm{Spec}(W_{0mm}(\omega))$ are not interlaced (i.e., they are separated by a real point p such that one of them lies below and the other above p) and it holds

$$\operatorname{dist}\left(\mathrm{Spec}(k\omega-\Delta_{mn}+W_{0nn}(\omega)),\,\mathrm{Spec}(W_{0mm}(\omega))\right)\geq\psi(k,n,m).$$

Proof. We distinguish two cases. If $k\neq 0$ then

$$|k\omega-\Delta_{mn}|=|k|\left|\omega-\frac{\Delta_{mn}}{k}\right|\geq\frac{7}{18}\,J|k|$$

since, by assumption,

$$\frac{\Delta_{mn}}{k}-\omega\in\,]-\infty,\frac{1}{2}\,J-\frac{8}{9}\,J\,]\cup[\,2J-\frac{9}{8}\,J,+\infty[\,.$$

So the distance may be estimated from below by

$$\frac{7}{18}\,J|k|-\|W_{0nn}(\omega)\|-\|W_{0mm}(\omega)\|\geq\frac{7}{18}\,J\left(|k|-\frac{1}{2}\right).$$

If $k=0$ then a lower bound to the distance is simply given by

$$\Delta_0-\|W_{0nn}(\omega)\|-\|W_{0mm}(\omega)\|\geq\frac{1}{2}\,\Delta_0.$$

□

Next we specify the way we shall construct the decreasing sequence of sets $\{\Omega_s\}_{s=0}^{\infty}$. Let $\Omega_0=[\,\frac{8}{9}J,\frac{9}{8}J\,]$. If $W_s\in\mathfrak{X}_s$ has been already defined then we introduce $\Omega_{s+1}\subset\Omega_s$ as the set of (W_s,ψ_s)–non-resonant frequencies. Recall that the real Banach space $\mathfrak{X}_s$ is determined by the choice of data φ_s, E_s and Ω_s, as explained in Section 5.

As a next step let us consider , for $s \in \mathbb{Z}_+$, $\omega \in \Omega_{s+1}$ and a non-diagonal index (k,n,m), a commutation equation,

$$(k\omega - \Delta_{mn} + (W_s)_{0nn}(\omega))X - X\,(W_s)_{0mm}(\omega) = Y, \tag{58}$$

with an unknown $X \in \mathcal{B}(\mathcal{H}_m, \mathcal{H}_n)$ and a right hand side $Y \in \mathcal{B}(\mathcal{H}_m, \mathcal{H}_n)$. Since ω is (W_s, ψ_s)–non-resonant the spectra $\mathrm{Spec}(k\omega - \Delta_{mn} + (W_s)_{0nn}(\omega))$ and $\mathrm{Spec}((W_s)_{0mm}(\omega))$ don't intersect and so a solution X exists and is unique. This way one can introduce a linear mapping

$$(\Gamma_s)_{knm}(\omega) : \mathcal{B}(\mathcal{H}_m, \mathcal{H}_n) \to \mathcal{B}(\mathcal{H}_m, \mathcal{H}_n)$$

such that $X = (\Gamma_s)_{knm}(\omega)Y$ solves (58). Moreover, according to Appendix A,

$$\|(\Gamma_s)_{knm}(\omega)\| \leq \frac{(\min\{M_m, M_n\})^{1/2}}{\psi(k,n,m)} \tag{59}$$

in the general case, and provided the spectra $\mathrm{Spec}(k\omega - \Delta_{mn} + (W_s)_{0nn}(\omega))$ and $\mathrm{Spec}((W_s)_{0mm}(\omega))$ are not interlaced it even holds that

$$\|(\Gamma_s)_{knm}(\omega)\| \leq \frac{1}{\psi(k,n,m)}\,. \tag{60}$$

From the uniqueness it is clear that $\mathrm{Ker}((\Gamma_s)_{knm}(\omega)) = 0$.
We extend the definition of $(\Gamma_s)_{knm}$ to diagonal indices by letting $(\Gamma_s)_{0nn}(\omega) = 0 \in \mathcal{B}(\mathcal{B}(\mathcal{H}_n, \mathcal{H}_n))$. This way we get an element

$$\Gamma_s \in \mathrm{Map}\left(\Omega_{s+1} \times \mathbb{Z} \times \mathbb{N} \times \mathbb{N}, \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}}{}^{\oplus} \mathcal{B}(\mathcal{B}(\mathcal{H}_m, \mathcal{H}_n))\right), \tag{61}$$

which naturally defines a linear mapping, denoted for simplicity by the same symbol, $\Gamma_s : {}^0\mathfrak{X}_s \to {}^0\mathfrak{X}_{s+1}$, according to the rule

$$\Gamma_s(Y)_{knm}(\omega) := (\Gamma_s)_{knm}(\omega)(Y_{knm}(\omega))\,.$$

Lemma 12. *Assume that for all non-diagonal indices (k,n,m) and $\omega, \omega' \in \Omega_{s+1}$, $\omega \neq \omega'$, it holds*

$$\|\partial(\Gamma_s)^{-1}_{knm}(\omega,\omega')\| \leq |k| + \frac{1}{2}, \tag{62}$$

if $\omega \in \Omega_{s+1}$ and (k,n,m) is a non-critical index then the spectra $\mathrm{Spec}(k\omega - \Delta_{mn} + (W_s)_{0nn}(\omega))$ and $\mathrm{Spec}((W_s)_{0mm}(\omega))$ are not interlaced and

$$\varphi_{s+1} \leq \min\left\{\frac{2}{3}\,\Delta_0, \frac{1}{6}\,J\right\}. \tag{63}$$

Then the following upper estimate on the norm of $\Gamma_s \in \mathcal{B}({}^0\mathfrak{X}_s, {}^0\mathfrak{X}_{s+1})$ holds true:

$$\|\Gamma_s\| \leq \frac{5}{2\,\varphi_{s+1}}\,.$$

Proof. To estimate $\|\Gamma_s\|$ we shall use relation (94) of Proposition 15 in Appendix B. Note that

$$\begin{aligned}\|\partial(\Gamma_s)_{knm}(\omega,\omega')\| &= \|(\Gamma_s)_{knm}(\omega)\,\partial(\Gamma_s)^{-1}_{knm}(\omega,\omega')\,(\Gamma_s)_{knm}(\omega')\|\\ &\le \|(\Gamma_s)_{knm}(\omega)\|\|(\Gamma_s)_{knm}(\omega')\|\left(|k|+\frac{1}{2}\right).\end{aligned} \tag{64}$$

If (k,n,m) is critical then we have, according to (59) and (56),

$$\|(\Gamma_s)_{knm}(\omega)\| \le \frac{1}{\varphi_{s+1}|k|^{1/2}}\,e^{\varrho_s|k|/2}$$

and consequently

$$\begin{aligned}&e^{-\varrho_s|k|}\left(\|(\Gamma_s)_{knm}(\omega)\|+\varphi_{s+1}\|\partial(\Gamma_s)_{knm}(\omega,\omega')\|\right)\\ &\quad\le e^{-\varrho_s|k|}\left(\frac{1}{\varphi_{s+1}|k|^{1/2}}\,e^{\varrho_s|k|/2}+\frac{|k|+\frac{1}{2}}{\varphi_{s+1}|k|}\,e^{\varrho_s|k|}\right)\\ &\quad\le \frac{1}{\varphi_{s+1}}\left(1+1+\frac{1}{2|k|}\right)\le\frac{5}{2\,\varphi_{s+1}}.\end{aligned}$$

If (k,n,m) is non-critical and $k\neq 0$ then we have, according to (60) and (56),

$$\|(\Gamma_s)_{knm}(\omega)\| \le \frac{18}{7J\left(|k|-\frac{1}{2}\right)}$$

and consequently

$$\begin{aligned}&e^{-\varrho_s|k|}\left(\|(\Gamma_s)_{knm}(\omega)\|+\varphi_{s+1}\|\partial(\Gamma_s)_{knm}(\omega,\omega')\|\right)\\ &\quad\le \frac{18}{7J\left(|k|-\frac{1}{2}\right)}\left(1+\varphi_{s+1}\frac{18\left(|k|+\frac{1}{2}\right)}{7J\left(|k|-\frac{1}{2}\right)}\right)\\ &\quad\le \frac{1}{\varphi_{s+1}}\frac{1}{6}\frac{36}{7}\left(1+\frac{1}{6}\frac{54}{7}\right)<\frac{2}{\varphi_{s+1}}.\end{aligned}$$

In the case when (k,n,m) is non-critical and $k=0$ one gets similarly $\|(\Gamma_s)_{knm}(\omega)\|\le 2/\Delta_0$ and

$$\begin{aligned}&e^{-\varrho_s|k|}\left(\|(\Gamma_s)_{knm}(\omega)\|+\varphi_{s+1}\|\partial(\Gamma_s)_{knm}(\omega,\omega')\|\right)\\ &\quad\le \frac{2}{\Delta_0}\left(1+\varphi_{s+1}\frac{1}{\Delta_0}\right)\le\frac{1}{\varphi_{s+1}}\frac{4}{3}\left(1+\frac{2}{3}\right)<\frac{5}{2\,\varphi_{s+1}}.\end{aligned}$$

□

Now we are able to specify the mappings Θ^s_u. Set

$$A_s=\Gamma_s\big((1-\mathcal{D}_s)(W_s-\iota_{s-1}(W_{s-1}))\big)\in{}^0\mathfrak{X}_{s+1}. \tag{65}$$

$W_s\in\mathfrak{X}_s$ satisfies (32) and thus one finds, when taking Hermitian adjoint of (58), that

$$((\Gamma_s)_{knm}(\omega)Y)^*=-(\Gamma_s)_{-k,m,n}(\omega)(Y^*)\,.$$

This implies that A_s obeys condition (46). The mappings Θ^s_u, $s<u$, are defined by equality (47) (see also the comment following the equality).

9. Proof of Theorem 1

We start from the specification of the sequences $\{\varphi_s\}$ and $\{E_s\}$,

$$\varphi_s = a\, s^\alpha q^{-rs} \text{ for } s \geq 1,\ E_s = q^{s+1} \text{ for } s \geq 0, \tag{66}$$

where $\alpha > 1$ and $q > 1$ are constants that are arbitrary except of the restrictions

$$q^r \geq e^\alpha \quad \text{and} \quad q^{-r}\zeta(\alpha) \leq 3\,\ln 2 \tag{67}$$

(ζ stands for the Riemann zeta function), and

$$a = 45\, e\, q^{2r} \epsilon_V. \tag{68}$$

For example, $\alpha = 2$ and $q^r = e^2$ will do. The value of $\varphi_0 \geq \varphi_1 = a\, q^{-r}$ doesn't influence the estimates which follow, and we automatically have $E_{-1} = 1$ (this is a convenient convention). Condition $r\,\ln(q) \geq \alpha$ guarantees that the sequence $\{\varphi_s\}$ is decreasing. Note also that

$$\varrho_s = \frac{1}{E_s} - \frac{1}{E_{s+1}} = \left(1 - \frac{1}{q}\right) q^{-s-1}.$$

Another reason for the choice (66) and (68) is that the constants $A_\star$, $B_\star$ and $C_\star$, as defined in (38), obey assumption (39) of Proposition 7. Particularly, a constraint on the choice of $\{\varphi_s\}$ and $\{E_s\}$, namely $\sum_{s=0}^\infty 1/(\varphi_{s+1}(E_{s-1})^r) < \infty$, is imposed by requiring $B_\star$ to be finite. However this is straightforward to verify. Actually, the constants may now be expressed explicitly,

$$A_\star = \frac{5e\, q^{2r}}{a},\ B_\star = \frac{5e\, q^r}{a}\,\zeta(\alpha),\ C_\star = \frac{5e\, q^r}{a}\,,$$

and thus conditions (39) mean that

$$\epsilon_V \frac{5e\, q^r}{a}\,\zeta(\alpha) \leq \frac{1}{3}\,\ln 2,\ \epsilon_V \frac{5e\, q^{2r}}{a}\,\phi\left(\epsilon_V \frac{15e\, q^r}{a}\right) \leq \frac{1}{9}\,. \tag{69}$$

The latter condition in (69) is satisfied since the LHS is bounded from above by (c.f. (5))

$$\frac{1}{9}\,\phi\left(\frac{1}{3}\,q^{-r}\right) \leq \frac{1}{9}\,\phi\left(\frac{1}{3}\right) = 1 - \frac{2}{3}\,e^{1/3} < \frac{1}{9}\,.$$

Concerning the former condition, the LHS equals $q^{-r}\zeta(\alpha)/9$ and so it suffices to chose α and q so that (67) is fulfilled. An additional reason for the choice (66) will be explained later.

Let us now summarise the construction of the sequences $\{\mathfrak{X}_s\}$, $\{W_s\}$ and $\{\Theta_u^s\}_{s>u}$ which will finally amount to a proof of Theorem 1. Some more details were already given in Section 8. We set $\Omega_0 = [\,\frac{8}{9}J, \frac{9}{8}J\,]$ and $W_0 = V_0$. Recall that the cut-offs V_s of V were introduced in (35). In every step, numbered by $s \in \mathbb{Z}_+$, we assume that Ω_t and W_t, with $0 \leq t \leq s$, and A_t, with $0 \leq t \leq s-1$, have already been defined. The mappings Θ_u^t, with $u > t$, are given by $\Theta_u^t(X) = [\,\iota_{u,t+1}(A_t), X\,]$ provided $A_t \in {}^0\mathfrak{X}_{t+1}$ satisfies condition (46). We define $\Omega_{s+1} \subset \Omega_s$ as the set of (W_s, ψ_s)–non-resonant frequencies, with ψ_s introduced in (56). Consequently, the real Banach space $\mathfrak{X}_{s+1}$ is defined as well as its definition depends on the data Ω_{s+1}, φ_{s+1} and E_{s+1}. Then we are able to introduce an element Γ_s (in the sense of (61)) whose definition is based on equation (58) and which in turn determines a bounded operator $\Gamma_s \in \mathcal{B}({}^0\mathfrak{X}_s, {}^0\mathfrak{X}_{s+1})$ (with some abuse of notation). The element $A_s \in {}^0\mathfrak{X}_{s+1}$ is given by equality (65) and actually satisfies condition (46). Knowing W_t, $t \leq s$, and Θ^t_{s+1}, $t \leq s$, (which is equivalent to knowing A_t, $t \leq s$) one is able to evaluate the RHS of (8) defining the element W_{s+1}. Hence one proceeds one step further.

We choose $\epsilon_\star(r, \Delta_0, J)$ maximal possible so that

$$\frac{3e}{1-q^{-r}}\,\epsilon_\star(r,\Delta_0,J) \leq \min\left\{\frac{1}{4}\,\Delta_0, \frac{7}{72}\,J\right\} \tag{70}$$

and

$$45\,e\,q^r\epsilon_\star(r,\Delta_0,J) \leq \min\left\{\frac{2}{3}\,\Delta_0, \frac{1}{6}\,J\right\}\,. \tag{71}$$

We claim that this choice guarantees that the construction goes through. Basically this means that $\epsilon_V < \epsilon_\star(r, \Delta_0, J)$ is sufficiently small so that all the assumptions occurring in the preceding auxiliary results are satisfied in every step, with $s \in \mathbb{Z}_+$. This concerns assumption (57) of Lemma 11,

$$\|(W_s)_{0mm}(\omega)\| \leq \min\left\{\frac{1}{4}\,\Delta_0, \frac{7}{72}\,J\right\}, \quad \forall\omega \in \Omega_s,\ \forall m \in \mathbb{N}, \tag{72}$$

assumption (54) of Lemma 10,

$$\|\partial(W_s)_{0mm}(\omega,\omega')\| \leq \frac{1}{4}, \quad \forall\omega,\omega' \in \Omega_s,\ \omega \neq \omega',\ \forall m \in \mathbb{N}, \tag{73}$$

assumptions (62) and (63) of Lemma 12,

$$\|\partial(\Gamma_s)^{-1}_{knm}(\omega,\omega')\| \leq |k| + \frac{1}{2}, \quad \forall(k,n,m),\ \forall\omega,\omega' \in \Omega_s,\ \omega \neq \omega', \tag{74}$$

and

$$\varphi_{s+1} \leq \min\left\{\frac{2}{3}\,\Delta_0, \frac{1}{6}\,J\right\}, \tag{75}$$

and assumption (50) of Proposition 9,

$$\|A_{s-1}\| \leq \frac{5}{2\varphi_s}\,\|W_{s-1} - \iota_{s-2}(W_{s-2})\|\,. \tag{76}$$

We can immediately do some simplifications. As the sequence $\{\varphi_s\}$ is non-increasing condition (75) reduces to the case $s = 0$. Since $\varphi_1 = 45\,e\,q^r\epsilon_V$ the upper bound (71) implies (75).

Note also that (74) is a direct consequence of (73). Actually, one deduces from the definition of $(\Gamma_s)_{knm}(\omega)$ (based on equation (58)) that, $\forall Y \in \mathcal{B}(\mathcal{H}_m, \mathcal{H}_n)$,

$$(\Gamma_s)^{-1}_{knm}(\omega)Y = (k\omega - \Delta_{mn} + (W_s)_{0nn}(\omega))Y - Y(W_s)_{0mm}(\omega)\,.$$

Hence

$$\partial(\Gamma_s)^{-1}_{knm}(\omega,\omega')Y = (k + \partial(W_s)_{0nn}(\omega,\omega'))\,Y - Y\,\partial(W_s)_{0mm}(\omega,\omega')$$

and, assuming (73),

$$\|\partial(\Gamma_s)^{-1}_{knm}(\omega,\omega')\| \leq |k| + \|\partial(W_s)_{0nn}(\omega,\omega'))\| + \|\partial(W_s)_{0mm}(\omega,\omega')\| \leq |k| + \frac{1}{2}\,.$$

Let us show that in every step, with $s \in \mathbb{Z}_+$, conditions (72), (73) and (76) are actually fulfilled. For $s = 0$, condition (76) is empty and condition (73) is obvious since $W_0 = V_0$

doesn't depend on ω. Condition (72) is obvious as well due to assumption (70) and the fact that $\|(W_0)_{0mm}(\omega)\| = \|(V_0)_{0mm}\| \le \epsilon_V$.
Assume now that $t \in \mathbb{Z}_+$ and conditions (72), (73) and (76) are satisfied in each step $s \le t$. Recall that in (40) we have set $F_s = 5/\varphi_{s+1}$ and $v_s = e\,\epsilon_V/(E_{s-1})^r$. We also keep the notation $w_s = \|W_s - \iota_{s-1}(W_{s-1})\|_s$, with the convention $W_{-1} = 0$.
We start with condition (76). Using the induction hypothesis, Lemma 11 and Lemma 12 one finds that $\|\Gamma_t\| \le F_t/2$ and so $\|A_t\| \le \|\Gamma_t\|\|W_t - \iota_{t-1}(W_{t-1})\| \le F_t w_t/2$ (c.f. (65) and (4)).
By the induction hypothesis and the just preceding step, $\|A_s\| \le F_s w_s$ for all $s \le t$. As we already know the constants $A_\star$, $B_\star$ and $C_\star$ fulfil (39) and so the quantities A, B and C given by $A = \epsilon_V A_\star$, $B = \epsilon_V B_\star$ and $C = \epsilon_V C_\star$ (c.f. (41)) obey (42) and consequently inequality (15) with $d = 3$. By the very choice of A, B and C (c.f. (38) and (40)) the quantities also obey relations (12), (13) and (14). This means that all assumptions of Proposition 3 are fulfilled for $s \le t$ (recall that $\|\Theta_u^s\| \le 2\|A_s\|$). One easily finds that the conclusion of Proposition 3, namely $w_s \le d\,v_s$, holds as well for all s, $s \le t+1$. Clearly, $\|(W_s)_{0mm}(\omega)\| \le \|W_s\|_s$ for all s, and

$$\|W_{t+1}\|_{t+1} \le \sum_{s=0}^{t+1} w_s \le 3\sum_{s=0}^{\infty} v_s = 3e\,\epsilon_V \sum_{s=0}^{\infty} q^{-rs} = \frac{3e}{1-q^{-r}}\,\epsilon_V.$$

By (70) we conclude that (72) is true for $s = t+1$.
Finally, using once more that $w_s \le 3v_s$ for $s \le t+1$,

$$\begin{aligned}\|\partial(W_{t+1})_{0mm}(\omega,\omega')\| &\le \sum_{s=0}^{t+1} \|\partial\big(W_s - \iota_{s-1}(W_{s-1})\big)_{0mm}(\omega,\omega')\| \\ &\le \sum_{s=0}^{t+1} \frac{1}{\varphi_s}\,\|W_s - \iota_{s-1}(W_{s-1})\|_s \\ &\le \sum_{s=0}^{\infty} \frac{3v_s}{\varphi_{s+1}}\,.\end{aligned}$$

However, the last sum equals (c.f. (40) and (42))

$$\frac{3}{5}\sum_{s=0}^{\infty} F_s v_s = \frac{3}{5}\,B \le \frac{1}{5}\,\ln 2 < \frac{1}{4}\,.$$

This verifies (73) for $s = t+1$ and hence the verification of conditions (72), (73) and (76) is complete.
Set, as before, $\Omega_\infty = \bigcap_{s=0}^{\infty} \Omega_s$. Next we are going to estimate the Lebesgue measure of Ω_∞,

$$|\Omega_\infty| = |\Omega_0| - |\Omega_0 \setminus \Omega_\infty| = \frac{17}{72}\,J - \sum_{s=0}^{\infty} |\Omega_s \setminus \Omega_{s+1}| = \frac{17}{72}\,J - \sum_{s=0}^{\infty} |\Omega_s^{\mathrm{bad}}|\,.$$

Recalling Lemma 10 jointly with Lemma 11 showing that the assumptions of Lemma 10 are satisfied, and the explicit form of ψ (56) we obtain

$$\begin{aligned}
|\Omega_s^{\mathrm{bad}}| &\leq 8\varphi_{s+1} \sum_{\substack{m,n\in\mathbb{N},\\ \Delta_{mn}>\frac{1}{2}J}} \mu_{mn} \sum_{\substack{k\in\mathbb{N},\\ \max\{1,\frac{\Delta_{mn}}{2J}\}<k<\frac{2\Delta_{mn}}{J}}} k^{-1/2} e^{-\varrho_s k/2} \\
&\leq 8\varphi_{s+1} \sum_{\substack{m,n\in\mathbb{N},\\ \Delta_{mn}>\frac{1}{2}J}} \mu_{mn} \frac{2\Delta_{mn}}{J} \left(\frac{\Delta_{mn}}{2J}\right)^{-1/2} e^{-\varrho_s \Delta_{mn}/4J} \\
&= 32\,(2J)^{\sigma} \varphi_{s+1} \sum_{\substack{m,n\in\mathbb{N},\\ \Delta_{mn}>\frac{1}{2}J}} \frac{\mu_{mn}}{(\Delta_{mn})^{\sigma}} \left(\frac{\Delta_{mn}}{2J}\right)^{\sigma+\frac{1}{2}} e^{-\varrho_s \Delta_{mn}/4J} \\
&\leq 32\,2^{\sigma} \varphi_{s+1} \left(\frac{2\sigma+1}{e\varrho_s}\right)^{\sigma+\frac{1}{2}} \Delta_{\sigma}(J)
\end{aligned}$$

where we have used that if $\alpha>0$ and $\beta>0$ then $\sup_{x>0} x^{\alpha} e^{-\beta x} = (\frac{\alpha}{e\beta})^{\alpha}$. To complete the estimate we need that the sum $\sum_{s=0}^{\infty} \varphi_{s+1}/(\varrho_s)^{\sigma+\frac{1}{2}}$ should be finite which imposes another restriction on the choice of $\{\varphi_s\}$ and $\{E_s\}$. With our choice (66) this is guaranteed by the condition $r>\sigma+\frac{1}{2}$ since in that case

$$\sum_{s=0}^{\infty} \frac{\varphi_{s+1}}{(\varrho_s)^{\sigma+\frac{1}{2}}} = \frac{a}{\left(1-\frac{1}{q}\right)^{\sigma+\frac{1}{2}}} \sum_{s=0}^{\infty} (s+1)^{\alpha} q^{-(r-\sigma-\frac{1}{2})(s+1)} < \infty\,.$$

Hence

$$|\Omega_\infty| \geq \frac{17}{72}\,J - \delta_1(\sigma,r)\,\Delta_{\sigma}(J)\,\epsilon_V \tag{77}$$

where

$$\delta_1(\sigma,r) = 1440\,e\,q^{2r} 2^{\sigma} \left(\frac{2\sigma+1}{\left(1-\frac{1}{q}\right)e}\right)^{\sigma+\frac{1}{2}} \mathrm{Li}_{-\alpha}(q^{-r+\sigma+\frac{1}{2}}) \tag{78}$$

Here $\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} z^k/k^n$ ($|z|<1$) is the polylogarithm function. This shows (3).
To finish the proof let us assume that $\omega\in\Omega_\infty$. We wish to apply Proposition 9. Going through its assumptions one finds that it only remains to make a note concerning equality (48). In fact, this equality is a direct consequence of the construction of $A_s \in {}^0\mathfrak{X}_{s+1}$. Actually, by the definition of A_s (c.f. (65)), $A_s = \Gamma_s\big((1-\mathcal{D}_s)(W_s - \iota_{s-1}(W_{s-1}))\big)$, which means that for any $\omega\in\Omega_{s+1}$ and all indices (k,n,m),

$$\begin{aligned}
&\big(k\omega - \Delta_{mn} + (W_s)_{0nn}(\omega)\big)(A_s)_{knm}(\omega) - (A_s)_{knm}(\omega)(W_s)_{0mm}(\omega) \\
&\qquad\qquad = \big((1-\mathcal{D}_s)(W_s - \iota_{s-1}(W_{s-1}))\big)_{knm}(\omega).
\end{aligned} \tag{79}$$

On the other hand, by the definition of Θ_u^s (c.f. (47)) and the definition of $\mathcal{D}_s$ (c.f (33)), and since $\omega\in\Omega_\infty$, it holds true that, $\forall u,\ u>s$,

$$\begin{aligned}
\Theta_u^s(\iota_{us}\mathcal{D}_s(W_s))_{knm}(\omega) &= \big([\,\iota_{u,s+1}(A_s), \iota_{us}\mathcal{D}_s(W_s)\,]\big)_{knm}(\omega) \\
&= (A_s)_{knm}(\omega)(W_s)_{0mm} - (W_s)_{0nn}(A_s)_{knm}(\omega).
\end{aligned} \tag{80}$$

A combination of (79) and (80) gives (48). We conclude that according to Proposition 9 the operator $\mathbf{K}+\mathbf{V}$ is unitarily equivalent to $\mathbf{K}+\mathbf{D}(\mathbf{W})$ and hence has a pure point spectrum. This concludes the proof of Theorem 1.

10. Concluding remarks

The backbone of the proof of Theorem 1 forms an iterative procedure loosely called here and elsewhere the quantum KAM method. One of the improvements attempted in the present paper was a sort of optimalisation of this method, particularly from the point of view of assumptions imposed on the regularity of the perturbation V. In this final section we would like to briefly discuss this feature by comparing our presentation to an earlier version of the method. We shall refer to paper [9] but the main points of the discussion apply as well to other papers including the original articles [5], [6] where the quantum KAM method was established. For the sake of illustration we use a simple but basic model: $H=\sum_{m\in\mathbb{N}} m^{1+\alpha}Q_m$, i.e., $h_m=m^{1+\alpha}$, with $0<\alpha\leq 1$, and $\dim Q_m=1$; thus $\mu_{mn}=1$ and any $\sigma>1/\alpha$ makes $\Delta_\sigma(J)$ finite. The perturbation V is assumed to fulfill (34) for a given $r\geq 0$.

According to Theorem 1, r is required to satisfy $r>\sigma+1/2$ which may be compared to reference [9, Theorem 4.1] where one requires

$$r>\mathbf{r}_1=4\sigma+6+\left[\frac{(4\sigma+6)\sigma}{1+\sigma}\right]+1. \tag{81}$$

The reason is that the procedure is done in two steps in the earlier version; in the first step preceding the iterative procedure itself the so-called adiabatic regularisation is applied on V in order to achieve a regularity in time and "space" (by the spatial part one means the factor $\mathcal{H}$ in $\mathcal{K}=L^2([\,0,T\,],dt)\otimes\mathcal{H}$) of the type

$$\exists r_1,r_2>\mathbf{r}_2=4\sigma+6,\quad \sup_{knm}|k|^{r_1}|n-m|^{r_2}|V_{knm}|<\infty. \tag{82}$$

The adiabatic regularisation brings in the summand $\left[\frac{(4\sigma+6)\sigma}{1+\sigma}\right]+1$. In the present version both the adiabatic regularisation and condition (82) are avoided. This is related to the choice of the norm in the auxiliary Banach spaces $\mathfrak{X}_s$,

$$\|X\|_s=\sup_{\omega\neq\omega'}\sup_n\sum_{k,m}F_s(k,n,m)\left(|X_{knm}(\omega)|+\varphi_s|\partial X_{knm}(\omega,\omega')|\right).$$

In the earlier version the weights were chosen as $F_s(k,n,m):=\exp((|k|+|n-m|)/E_s)$ in order to compensate small divisors occurring in each step of the iterative method. A more careful control of the small divisors in the present version allows less restrictive weights, namely $F_s(k,n,m)=\exp(|k|/E_s)$. In more detail, indices labelling the small divisors are located in a critical subset of the lattice $\mathbb{Z}\times\mathbb{N}\times\mathbb{N}$. Definition (51) of the critical indices implies a simple estimate,

$$|k|\leq|k|+|n-m|\leq|k|+|\Delta_{mn}|\leq|k|(1+2J),$$

which explains why we effectively have, in the present version, $r_2=0$.

The second remark concerns Diophantine-like estimates of the small divisors governed by the sequence $\{\psi_s\}$. A bit complicated definition (56) is caused by the classification of the indices into critical and non-critical ones. However only the critical indices are of importance in this context and thus we can simplify, for the purpose of this discussion, the definition of ψ_s to

$$\psi_s = \gamma_s |k|^{1/2} e^{-\varrho_s |k|/2}, \quad \varphi_{s+1} \geq \gamma_s > 0.$$

Let us compare it to the choice made in [9], namely $\psi_s = \gamma_s |k|^{-\sigma}$. The factors γ_s then occur in some key estimates; let us summarise them. The norm of the operators $\Gamma_s : \mathfrak{X}_s \to \mathfrak{X}_{s+1}$ are estimated as

$$\|\Gamma_s\| \leq \text{const}\ \frac{\varphi_{s+1}}{\gamma_s^2}$$

(this is shown in Lemma 12 but note that in this lemma we have set $\gamma_s = \varphi_{s+1}$). Another important condition is the convergence of the series

$$B_\star = \text{const} \sum_{s=0}^{\infty} \frac{\varphi_{s+1}}{\gamma_s^2 (E_{s-1})^r} < \infty$$

(c.f. (38) but there again $\gamma_s = \varphi_{s+1}$). Finally, the measure of the set of resonant frequencies, $|\cup_s \Omega_s^{\text{bad}}|$, is estimated by

$$\sum_{s=0}^{\infty} |\Omega_s^{\text{bad}}| \leq \text{const} \sum_{s=0}^{\infty} \frac{\gamma_s}{\varrho_s^{\sigma+\frac{1}{2}}} < \infty, \quad \varrho_s = \frac{1}{E_s} - \frac{1}{E_{s+1}}$$

(shown in the part of the proof of Theorem 1 preceding relation (77)). We recall that E_s denotes the width of the truncation of the perturbation V at step s of the algorithm (c.f. (35)). These conditions restrict the choice of the sequences $\{E_s\}$ and $\{\gamma_s\}$ which may also be regarded as parameters of the procedure. Specification (66) of these parameters, with $\gamma_s = \varphi_{s+1}$, can be compared to a polynomial behaviour of E_s and γ_s in the variable s in [9] where one sets $\varphi_{s+1} \equiv 1$ and

$$E_s = \text{const}\,(s+1)^{\nu-1}, \quad \nu > 2, \qquad \gamma_s = \text{const}\,(s+1)^{-\mu}, \quad \mu > 1.$$

The latter definition finally leads to the bound on the order of regularity of V

$$r > \frac{(2\sigma+1)\nu + 3}{\nu - 1}.$$

Thus in that case the bound varies from $r > 4\sigma + 5$ (for $\nu \to 2+$; this contributes to $\mathbf{r}_1$ in (81)) to $r > 2\sigma + 1$ ($\nu \to +\infty$). This shows why we have chosen here to truncate with exponential E_s, see (66).

In the last remark let us mention a consequence of the equality $\gamma_s = \varphi_{s+1}$. The conditions for convergence of $B_\star$ and $\cup_s \Omega_s^{\text{bad}}$ become (notice that $\varrho_s = \text{const}\,/E_s$)

$$\sum_s \frac{1}{\varphi_{s+1}(E_{s-1})^r} < \infty \quad \text{and} \quad \sum_s \varphi_{s+1} E_s^{\sigma+\frac{1}{2}} < \infty$$

and are fulfilled for $r > \sigma + \frac{1}{2}$. There is however a drawback with this choice. Notice the role the coefficients φ_s play in the definition (31) of the norm $\|\cdot\|_s$. Since $\varphi_s \to 0$ as $s \to \infty$ one looses the control of the Lipschitz regularity in ω in the limit of the iterative procedure. This means that we have no information about the regularity of the eigenvectors and the eigenvalues of $\mathbf{K} + \mathbf{V}$ with respect to ω. With $r > 2\sigma + 1$ we could have taken $\varphi_{s+1} = 1$ and obtained that these eigenvalues and vectors are indeed Lipschitz in ω.

Appendix A. Commutation equation

Suppose that $\mathfrak{X}$ and $\mathfrak{Y}$ are Hilbert spaces, $\dim\mathfrak{X} < \infty$, $\dim\mathfrak{Y} < \infty$, $A \in \mathcal{B}(\mathfrak{Y})$, $B \in \mathcal{B}(\mathfrak{X})$, both A and B are self-adjoint, and $V \in \mathcal{B}(\mathfrak{X},\mathfrak{Y})$. If γ is a simple closed and positively oriented curve in the complex plane such that $\operatorname{Spec}(A)$ lies in the domain encircled by γ while $\operatorname{Spec}(B)$ lies in its complement then the equation

$$AW - WB = V \tag{83}$$

has a unique solution $W \in \mathcal{B}(\mathfrak{X},\mathfrak{Y})$ given by

$$W = \frac{1}{2\pi\imath}\oint_\gamma (A-z)^{-1}V(B-z)^{-1}dz\,. \tag{84}$$

The verification is straightforward.
Denote $M_1 = \dim\mathfrak{X}$, $M_2 = \dim\mathfrak{Y}$. We shall need the following two estimates on the norm of $X \in \mathcal{B}(\mathfrak{X},\mathfrak{Y})$:

$$\|X\|^2 \;\leq\; \sum_{i=1}^{M_2}\sum_{j=1}^{M_1}|X_{ij}|^2 = \operatorname{Tr}X^*X \quad (\text{Hilbert}-\text{Schmidt norm}), \tag{85}$$

$$\|X\|^2 \;\geq\; \max\left\{\max_{1\leq i\leq M_2}\sum_{j=1}^{M_1}|X_{ij}|^2,\ \max_{1\leq j\leq M_1}\sum_{i=1}^{M_2}|X_{ij}|^2\right\}, \tag{86}$$

where (X_{ij}) is a matrix of X expressed with respect to any orthonormal bases in $\mathfrak{X}$ and $\mathfrak{Y}$.
If $\sup\operatorname{Spec}(A) < \inf\operatorname{Spec}(B)$ or $\sup\operatorname{Spec}(B) < \inf\operatorname{Spec}(A)$ we shall say that $\operatorname{Spec}(A)$ and $\operatorname{Spec}(B)$ are not interlaced.

Proposition 13. *If* $\operatorname{Spec}(A)$ *and* $\operatorname{Spec}(B)$ *are not interlaced then*

$$\|W\| \leq \frac{\|V\|}{\operatorname{dist}(\operatorname{Spec}(A),\operatorname{Spec}(B))}\,,$$

otherwise, if $\operatorname{Spec}(A)$ *and* $\operatorname{Spec}(B)$ *don't intersect but are interlaced,*

$$\|W\| \leq (\min\{\dim\mathfrak{X},\dim\mathfrak{Y}\})^{1/2}\frac{\|V\|}{\operatorname{dist}(\operatorname{Spec}(A),\operatorname{Spec}(B))}\,.$$

Proof. (1) If $d = \inf\operatorname{Spec}(B) - \sup\operatorname{Spec}(A) > 0$ then, after a usual limit procedure, we can choose for the integration path in (84) the line which is parallel to the imaginary axis and intersects the real axis in the point $x_0 = (\sup\operatorname{Spec}(A) + \inf\operatorname{Spec}(B))/2$. So

$$\begin{aligned}\|W\|^2 &\leq \frac{1}{2\pi}\int_{-\infty}^{\infty}\|(A-x_0-\imath s)^{-1}\|\|V\|\|(B-x_0-\imath s)^{-1}\|\,ds\\ &= \frac{\|V\|}{2\pi}\int_{-\infty}^{\infty}\frac{ds}{\left(\frac{d}{2}\right)^2+s^2}\\ &= \frac{\|V\|}{d}\,.\end{aligned}$$

(2) In the interlaced case we choose orthonormal bases in $\mathfrak{X}$ and $\mathfrak{Y}$ so that A and B are diagonal, $A = \operatorname{diag}(a_1,\ldots,a_{M_2})$ and $B = (b_1,\ldots,b_{M_1})$. For brevity let us denote

dist(Spec(A), Spec(B)) by d. Then $W_{ij} = V_{ij}/(a_i - b_j)$, and we can use (85), (86) to estimate

$$\begin{aligned}\|W\|^2 &\leq \sum_{i=1}^{M_2}\sum_{j=1}^{M_1}\left|\frac{V_{ij}}{a_i - b_j}\right|^2 \leq \sum_{i=1}^{M_2}\sum_{j=1}^{M_1}\frac{|V_{ij}|^2}{d^2} \\ &\leq \sum_{i=1}^{M_2}\frac{\|V\|^2}{d^2} = M_2\frac{\|V\|^2}{d^2}\,.\end{aligned}$$

Symmetrically, $\|W\| \leq M_1^{1/2}\|V\|/d$, and the result follows. □

Appendix B. Choice of a norm in a Banach space

Let

$$\mathcal{H} = \sum_{n\in\mathbb{N}}^{\oplus}\mathcal{H}_n$$

be a decomposition of a Hilbert space into a direct sum of mutually orthogonal subspaces, and $\Omega \subset \mathbb{R}$. To any couple of positive real numbers, φ and E, we relate a subspace

$$\mathfrak{A} \subset L^\infty\left(\Omega\times\mathbb{Z}\times\mathbb{N}\times\mathbb{N}, \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}}^{\oplus}\mathcal{B}(\mathcal{H}_m,\mathcal{H}_n)\right)$$

formed by those elements $\mathcal{V}$ which satisfy

$$\mathcal{V}_{knm}(\omega) \in \mathcal{B}(\mathcal{H}_m,\mathcal{H}_n)$$

and have finite norm

$$\|\mathcal{V}\| = \sup_{\substack{\omega,\omega'\in\Omega\\ \omega\neq\omega'}}\sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{N}}\left(\|\mathcal{V}_{knm}(\omega)\| + \varphi\,\|\partial\mathcal{V}_{knm}(\omega,\omega')\|\right)e^{|k|/E} \tag{87}$$

where ∂ stands for the difference operator

$$\partial\mathcal{V}(\omega,\omega') = \frac{\mathcal{V}(\omega) - \mathcal{V}(\omega')}{\omega - \omega'}\,.$$

Note that the difference operator obeys the rule

$$\partial(\mathcal{U}\mathcal{V})(\omega,\omega') = \partial\mathcal{U}(\omega,\omega')\,\mathcal{V}(\omega') + \mathcal{U}(\omega)\partial\mathcal{V}(\omega,\omega')\,. \tag{88}$$

Proposition 14. *The norm in $\mathfrak{A}$ is an algebra norm with respect to the multiplication*

$$(\mathcal{U}\mathcal{V})_{knm}(\omega) = \sum_{\ell\in\mathbb{Z}}\sum_{p\in\mathbb{N}}\mathcal{U}_{k-\ell,n,p}(\omega)\,\mathcal{V}_{\ell pm}(\omega)\,. \tag{89}$$

Proof. We have to show that

$$\|\mathcal{U}\mathcal{V}\| \leq \|\mathcal{U}\|\|\mathcal{V}\|\,. \tag{90}$$

For brevity let us denote (in this proof)

$$\begin{aligned}
\mathcal{X}_p(\omega) &= \sum_{\ell\in\mathbb{Z}}\sum_{m\in\mathbb{N}} \|\mathcal{V}_{\ell pm}(\omega)\|\, e^{|\ell|/E}, \\
\partial\mathcal{X}_p(\omega,\omega') &= \sum_{\ell\in\mathbb{Z}}\sum_{m\in\mathbb{N}} \|\partial\mathcal{V}_{\ell pm}(\omega,\omega')\|\, e^{|\ell|/E}.
\end{aligned}$$

Here $\partial\mathcal{X}$ is an "inseparable" symbol (which this time doesn't have the meaning ∂ of $\mathcal{X}$). It holds

$$\begin{aligned}
&\sum_k\sum_m \|(\mathcal{U}\mathcal{V})_{knm}(\omega)\|\, e^{|k|/E} \\
&\quad\le \sum_k\sum_m\sum_\ell\sum_p \|\mathcal{U}_{k-\ell,n,p}(\omega)\|\, e^{|k-\ell|/E}\|\mathcal{V}_{\ell pm}(\omega)\|\, e^{|\ell|/E} \\
&\quad= \sum_k\sum_m\sum_\ell\sum_p \|\mathcal{U}_{knp}(\omega)\|\, e^{|k|/E}\|\mathcal{V}_{\ell pm}(\omega)\|\, e^{|\ell|/E} \\
&\quad= \sum_k\sum_p \|\mathcal{U}_{knp}(\omega)\|\, e^{|k|/E}\mathcal{X}_p(\omega)\,.
\end{aligned}$$

Similarly, using (88),

$$\begin{aligned}
\sum_k\sum_m \|\partial(\mathcal{U}\mathcal{V})_{knm}(\omega)\|\, e^{|k|/E} &\le \sum_k\sum_m\sum_\ell\sum_p \big(\|\mathcal{U}_{knp}(\omega)\|\, e^{|k|/E}\|\partial\mathcal{V}_{\ell pm}(\omega,\omega')\|\, e^{|\ell|/E} \\
&\qquad +\|\partial\mathcal{U}_{knp}(\omega,\omega')\|\, e^{|k|/E}\|\mathcal{V}_{\ell pm}(\omega')\|\, e^{|\ell|/E}\big) \\
&= \sum_k\sum_p \big(\|\mathcal{U}_{knp}(\omega)\|\,\partial\mathcal{X}_p(\omega,\omega') \\
&\qquad +\|\partial\mathcal{U}_{knp}(\omega,\omega')\|\,\mathcal{X}_p(\omega')\big)\, e^{|k|/E}\,.
\end{aligned}$$

A combination of these two inequalities gives

$$\begin{aligned}
&\sum_k\sum_m \big(\|(\mathcal{U}\mathcal{V})_{knm}(\omega)\| + \varphi\,\|\partial(\mathcal{U}\mathcal{V})_{knm}(\omega,\omega')\|\big)\, e^{|k|/E} \\
&\quad\le \sum_k\sum_p \big(\|\mathcal{U}_{knp}(\omega)\|(\mathcal{X}_p(\omega)+\varphi\,\partial\mathcal{X}_p(\omega,\omega')) + \varphi\,\|\partial\mathcal{U}_{knp}(\omega,\omega')\|\,\mathcal{X}_p(\omega')\big)\, e^{|k|/E} \\
&\quad\le \sup_{\omega,\omega'}\sup_p(\mathcal{X}_p(\omega)+\varphi\,\partial\mathcal{X}_p(\omega,\omega'))\sum_k\sum_p(\|\mathcal{U}_{knp}(\omega)\| + \varphi\,\|\partial\mathcal{U}_{knp}(\omega,\omega')\|)e^{|k|/E} \\
&\quad= \|\mathcal{V}\|\sum_k\sum_p(\|\mathcal{U}_{knp}(\omega)\| + \varphi\,\|\partial\mathcal{U}_{knp}(\omega,\omega')\|)e^{|k|/E}\,.
\end{aligned}$$

To obtain (90) it suffices to apply $\sup_{\omega,\omega'}\sup_n$ to this inequality. □

Suppose now that two couples of positive real numbers, (φ_1, E_1) and (φ_2, E_2), are given and that it holds

$$\varrho = \frac{1}{E_1} - \frac{1}{E_2} \ge 0 \quad\text{and}\quad \varphi_2 \le \varphi_1\,. \tag{91}$$

Consequently, we have two Banach spaces, $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Furthermore, we suppose that there is given an element

$$\Gamma \in \mathrm{Map}\left(\Omega \times \mathbb{Z} \times \mathbb{N} \times \mathbb{N}, \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}}^{\oplus} \mathcal{B}(\mathcal{B}(\mathcal{H}_m, \mathcal{H}_n))\right), \tag{92}$$

such that for each couple $(\omega, k) \in \Omega \times \mathbb{Z}$ and each double index $(n, m) \in \mathbb{N}\times\mathbb{N}$, $\Gamma_{knm}(\omega)$ belongs to $\mathcal{B}(\mathcal{B}(\mathcal{H}_m, \mathcal{H}_n))$. Γ naturally determines a linear mapping, called for the sake of simplicity also Γ, from $\mathfrak{A}_1$ to $\mathfrak{A}_2$, according to the prescription

$$\Gamma(\mathcal{V})_{knm}(\omega) = \Gamma_{knm}(\omega)(\mathcal{V}_{knm}(\omega)). \tag{93}$$

Concerning the difference operator, in this case one can apply the rule

$$\partial\left(\Gamma(\mathcal{V})\right)(\omega, \omega') = \partial\Gamma(\omega, \omega')\left(\mathcal{V}(\omega')\right) + \Gamma(\omega)(\partial\mathcal{V}(\omega, \omega')).$$

Proposition 15. *The norm of $\Gamma : \mathfrak{A}_1 \to \mathfrak{A}_2$ can be estimated as follows,*

$$\|\Gamma\| \leq \sup_{\substack{\omega,\omega'\in\Omega\\ \omega\neq\omega'}} \sup_{k\in\mathbb{Z}} \sup_{(n,m)\in\mathbb{N}\times\mathbb{N}} e^{-\varrho|k|}\left(\|\Gamma_{knm}(\omega)\| + \varphi_2 \|\partial\Gamma_{knm}(\omega, \omega')\|\right). \tag{94}$$

Proof. Notice that, if convenient, one can interchange ω and ω' in $\|\partial\mathcal{U}(\omega, \omega')\|$. It holds

$$\begin{aligned}
&\sum_k \sum_m \left(\|\Gamma_{knm}(\omega)(\mathcal{V}_{knm}(\omega))\| + \varphi_2 \|\partial\left(\Gamma_{knm}(\mathcal{V}_{knm})\right)(\omega, \omega')\|\right) e^{|k|/E_2}\\
&\quad\leq \sum_k \sum_m \Big(\|\mathcal{V}_{knm}(\omega)\|(\|\Gamma_{knm}(\omega)\| + \varphi_2 \|\partial\Gamma_{knm}(\omega, \omega')\|)e^{-\varrho|k|}\\
&\qquad\quad +\varphi_2 \|\partial\mathcal{V}_{knm}(\omega, \omega')\| \, \|\Gamma_{knm}(\omega')\|e^{-\varrho|k|}\Big)\, e^{|k|/E_1}\\
&\quad\leq \sup_{\omega,\omega'} \sup_{k} \sup_{(n,m)} e^{-\varrho|k|}\left(\|\Gamma_{knm}(\omega)\| + \varphi_2 \|\partial\Gamma_{knm}(\omega, \omega')\|\right)\\
&\qquad\quad \times \sum_k \sum_m \left(\|\mathcal{V}_{knm}(\omega)\| + \varphi_1 \|\partial\mathcal{V}_{knm}(\omega, \omega')\|\right) e^{|k|/E_1}.
\end{aligned}$$

To finish the proof it suffices to apply $\sup_{\omega,\omega'} \sup_n$ to this inequality. ☐

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